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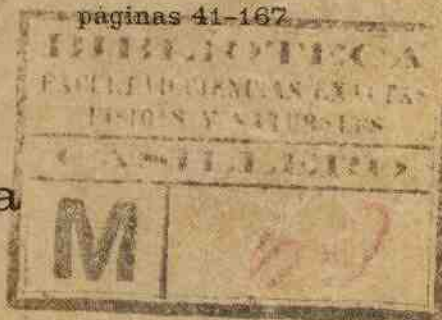


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A combinatorial inequality and its applications to L^2 -spaces*

BY M. COTLAR

In this paper we give an estimate of the norm for a certain class of operators T in L^2 -spaces. Since the composition of operators generally improves their norms, our idea is to decompose T into a sum $T = T_1 + \dots + T_N$, in such a way that $T_i T_j$ is very small when $|i - j|$ is great.

This estimate, together with the results of the two following papers, will permit us to unify the theory of Hilbert transforms and ergodic theorems.

1. Generalized integrals. Let $E = \{1, 2, \dots, N\}$ be a finite set of N elements, and μ a set function assigning to each subset $H \subset E$ a non-negative number $\mu(H) \geq 0$. $\mu(H)$ is not assumed to be additive, we only require that $H \subset H'$ implies $\mu(H) \leq \mu(H')$. We shall denote by $\varphi_H(i)$ the characteristic function of the set H : $\varphi_H(i) = 1$ if $i \in H$, and $\varphi_H(i) = 0$ otherwise.

Let $f(i) \geq 0$ be a function defined on E whose values are non-negative integers: $f(1) = \alpha_1, f(2) = \alpha_2, \dots, f(N) = \alpha_N$.

The function $f(i)$ admits a finite number of representations of the form $f(i) = \lambda_1 \varphi_{H_1}(i) + \dots + \lambda_n \varphi_{H_n}(i)$, where the λ_i are non-negative integers, while the sets H_i may overlap. For each such representation we form the sum $s = \lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n)$, and define the integral, or sum, of $f(i)$ with respect to $\mu(H)$, by

$$(1) \quad \sum f \Delta \mu = \text{Max } s = \text{Max } \{\lambda_1 \mu(H_1) + \dots + \lambda_n \mu(H_n)\}.$$

From the definition of $\sum f \Delta \mu$ it is clear that

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$$(2) \quad f = f_1 + f_2 \quad \text{implies} \quad \sum f \Delta \mu \geq \sum f_1 \Delta \mu + \sum f_2 \Delta \mu.$$

If μ is additive we obtain the ordinary definition of integral.

LEMMA 1. Let $f(1) = \alpha_1, \dots, f(N) = \alpha_N$, and let $\beta_1 \geq \beta_2 \geq \dots \geq \beta_N$ be the rearrangement in decreasing order of the sequence $\alpha_1, \alpha_2, \dots, \alpha_N$. If H_k denote the set where $f(i) \geq \beta_k$, then

$$\sum f \Delta \mu \geq \sum_{k=2}^N \beta_k (\mu(H_k) - \mu(H_{k-1})) + \beta_1 \mu(H_1).$$

PROOF. Since

$$(2') \quad f(i) = \beta_N \varphi_N + (\beta_{N-1} - \beta_N) \varphi_{N-1} + (\beta_{N-2} - \beta_{N-1}) \varphi_{N-2} + \dots$$

where φ_k stands for φ_{H_k} , it follows that

$$\begin{aligned} \sum f \Delta \mu &\geq \beta_N \mu(H_N) + (\beta_{N-1} - \beta_N) \mu(H_{N-1}) + \dots = \\ &= \beta_N \mu(H_N) + \sum_{k=2}^N (\beta_k - \beta_{k-1}) \mu(H_{k-1}) = \\ &= \sum_{k=2}^N \beta_k (\mu(H_k) - \mu(H_{k-1})) + \beta_1 \mu(H_1). \end{aligned}$$

Now we shall consider the particular measure $\mu(H)$ defined as follows: If $H = \{h_1, h_2, \dots, h_m\}$ and $h_1 < h_2 < \dots < h_m$, then

$$(3) \quad \mu(H) = 5(h_m - h_1) + (h_{m-1} - h_2) + (h_{m-2} - h_3) + \dots,$$

and if $H_0 = \{h_0\}$, then

$$(3a) \quad \mu(H_0) = 0.$$

From (3) we have that

$$(4) \quad \mu(H) \geq 5(h_m - h_1), \quad \text{and} \quad \mu(H) \leq m^2,$$

if $m > 1$.

LEMMA 2. Let $\mu(H)$ be the measure defined by (3), and $f(i)$ a function defined on $E = \{1, 2, \dots, N\}$. If $H = \{h_1, \dots, h_r\}$, ($h_1 < h_2 < \dots < h_r$), is the support of $f(i)$, that is the set where $f(i) \neq 0$, and if $f(h_1) = \alpha_1$, then

$$\sum f \Delta \mu \geq 5(h_r - h_1) + \sum_{i=1}^r i \beta_i - 2(\beta_1 + \beta_2),$$

where $\beta_1 \geq \beta_2 \geq \dots \geq \beta_r$ is the rearrangement in decreasing order of the sequence $\alpha_1, \alpha_2, \dots, \alpha_r$.

PROOF. Let $H' = \{h_1, h_r\}$, $H'' = E - H'$, $f_1 = \chi_{H'}$, $f_2 = f - f_1$, so that $f = f_1 + f_2$. Then

$$(5) \quad \sum f \Delta \mu \geq \sum f_1 \Delta \mu + \sum f_2 \Delta \mu \geq \mu(H') + \sum f_2 \Delta \mu = \\ = 5(h_r - h_1) + \sum f_2 \Delta \mu.$$

Let $\mu'(H)$ be the measure defined as follows: if $H = \{i_1, \dots, i_m\}$, then $\mu'(H) = m^2$ if $m > 1$, and $\mu'(H) = 0$ if $m = 1$. By (4)

$$\mu(H) \geq \mu'(H), \quad \sum f_2 \Delta \mu \geq \sum f_2 \Delta \mu'.$$

Let $f_2(i) = \alpha'_i$, and let $\beta'_1 \geq \beta'_2 \geq \dots$ be the rearrangement in decreasing order of the sequence $\alpha'_1, \alpha'_2, \dots$. If H_k is the set where $f_2(i) \geq \beta'_k$ then H_k contains $\geq k$ elements and

$$\mu'(H_k) \geq k^2, \text{ and } k^2 - (k-1)^2 \geq k+2, \text{ if } k \geq 3.$$

Applying lemma (2') and taking in account that $\beta'_i \geq \beta'_{i+2}$, we obtain

$$\sum f_2 \Delta \mu \geq \sum f_2 \Delta \mu' \geq \sum_k (\beta'_k - \beta'_{k+1}) k^2 \\ \geq \sum_{k=3}^r (k+2) \beta'_{i+2} \geq \sum_{k=1}^r k \beta'_k - 2\beta'_1 - 2\beta'_2.$$

This, together to (5), proves Lemma 2.

2. The main inequality. If $f(i)$ is defined on $E = \{1, 2, \dots, N\}$ and $f(i) = \alpha_i$, we shall write

$$f(i) = \begin{pmatrix} \alpha_1 \alpha_2 \dots \alpha_N \\ 1 \ 2 \dots N \end{pmatrix}, \quad \sum f \Delta \mu = \sum \begin{pmatrix} \alpha_1 \alpha_2 \dots \alpha_N \\ 1 \ 2 \dots N \end{pmatrix} \Delta \mu.$$

Let k be a fixed integer and $\lambda > 1$ a real number. Consider all the functions $f_i(i)$ defined on E such that $f_i(1) + f_i(2) + \dots + f_i(N) = k$ ($f_i(i) =$ non-negative integers), and for each such $f_i(i)$ form the number $\lambda^{-\sum f_i \Delta \mu}$. We shall give an estimate of the sum $S = \sum \lambda^{-\sum f_i \Delta \mu}$. More precisely:

LEMMA 3. Let $\mu(H)$ be the measure defined by (3), $E = \{1, 2, \dots, N\}$, N and k fixed integers, and $\lambda > 1$ a real number such that $\lg \lambda > 2^{1/k-1}$. Then

$$S = \sum_{\alpha_1 + \dots + \alpha_N = k} \frac{k!}{\alpha_1! \dots \alpha_N!} \lambda^{-\sum_{i=1}^N \binom{\alpha_i}{2}} \Delta_N$$

$$\leq \frac{\lambda^{2k} \cdot k \cdot k^{(k)/2} \cdot N}{(\lambda^2 - 1)^{2k} (\lambda - 1)^k}$$

PROOF. Consider a group of r elements $h_1 < h_2 < \dots < h_r \in E$ and r integers $\alpha_1, \dots, \alpha_r$, such that $\alpha_1 + \dots + \alpha_r = k$ and all the $\alpha_i > 0$. Since the support of the function $\binom{\alpha_1, \dots, \alpha_r}{h_1, \dots, h_r}$ is the set $(h_1, \dots, h_r)$, by Lemma 2

$$\lambda^{-\sum_{i=1}^r \binom{\alpha_i}{2}} \Delta_N \leq \frac{\lambda^{2k}}{\lambda^{2(h_r - h_1)} \lambda^{2\beta_1 + 2\beta_2 + \dots + 2\beta_r}}$$

where $\beta_1 \geq \beta_2 \geq \dots \geq \beta_r$ is the rearrangement in decreasing order of the sequence $\alpha_1, \dots, \alpha_r$.

Let us fix the number r , the elements $h_1, \dots, h_r$, and the numbers $\beta_1 \geq \dots \geq \beta_r$, and let $\Gamma(\beta_1, \dots, \beta_r)$ denote the set of all groups $\alpha_1, \dots, \alpha_r$, $\alpha_i = \beta_j$, $\alpha_1 + \dots + \alpha_r = k$, $\alpha_i \neq 0$. Then, since $\Gamma(\beta_1, \dots, \beta_r)$ contains at most $r!$ groups,

$$\sum_{\substack{(\alpha_1, \dots, \alpha_r) \in \Gamma(\beta_1, \dots, \beta_r) \\ h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum_{i=1}^r \binom{\alpha_i}{2}} \Delta_N$$

$$\leq r! \frac{\lambda^{2k}}{\lambda^{2(h_r - h_1)}} \frac{k!}{\alpha_1! \dots \alpha_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \left(\frac{1}{\lambda}\right)^{\beta_2} \dots \left(\frac{1}{\lambda}\right)^{\beta_r}.$$

Therefore, if we keep r and $h_1, \dots, h_r$ fixed and let the α_i vary under the condition $\alpha_1 + \dots + \alpha_r = k$, $\alpha_i \neq 0$, we obtain

$$(6) \quad \sum_{\substack{\alpha_1 + \dots + \alpha_r = k \\ \alpha_i \neq 0 \\ r, h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum_{i=1}^r \binom{\alpha_i}{2}} \Delta_N \leq$$

$$\frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \sum_{\beta_1 + \dots + \beta_r = k} \frac{k!}{\beta_1! \dots \beta_r!} \left(\frac{1}{\lambda}\right)^{\beta_1} \dots \left(\frac{1}{\lambda}\right)^{\beta_r}$$

$$= \frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \left(\frac{1}{\lambda} + \dots + \frac{1}{\lambda}\right)^k$$

$$\leq \frac{r! \lambda^{2k}}{\lambda^{2(h_r - h_1)}} \left(\frac{1}{\lambda - 1}\right)^k.$$

We give now another estimate of the left member of (6).
 Since $\lambda = 0, +2\alpha_1 + \dots + r\alpha_r \leq \lambda = (1+2+\dots+r) \leq (\sqrt{\lambda})^2$, we have

$$\begin{aligned}
 (7) \quad & \sum_{\substack{u_1+\dots+u_r=k \\ u_i \geq 0 \\ r, h_1, \dots, h_r \text{ fixed}}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum \binom{u_1, \dots, u_r}{h_1, \dots, h_r} \Delta_i} \\
 & \leq \frac{\lambda^{2k}}{\lambda^{5(h_r-h_1)}} \cdot \frac{1}{(\sqrt{\lambda})^{r^2}} \sum_{u_1+\dots+u_r=k} \frac{k!}{\alpha_1! \dots \alpha_r!} \\
 & = \frac{\lambda^{2k}}{\lambda^{5(h_r-h_1)}} \cdot \frac{r^k}{(\sqrt{\lambda})^{r^2}}
 \end{aligned}$$

If we keep h_1 and h_r fixed and let the $h_2, h_3, \dots, h_{r-1}$ vary under the condition $h_1 < h_2 < \dots < h_r$, we get at most $C_{h_r-h_1}^r$ sums of the form (6) or (7). Therefore, if S_r denotes the sum of all the terms of the form (6) or (7) where only r remained fixed, we obtain from (6) and (7) respectively:

$$\begin{aligned}
 (6a) \quad S_r &= \sum_{\substack{u_1+\dots+u_r=k \\ u_i \geq 0, r \text{ fixed} \\ 1 \leq h_1 < h_2 < \dots < h_r \leq N}} \frac{k!}{\alpha_1! \dots \alpha_r!} \lambda^{-\sum \binom{u_1, \dots, u_r}{h_1, \dots, h_r} \Delta_i} \\
 &\leq \sum_{h_1=1}^N \sum_{h_r=h_1+r}^N C_{h_r-h_1}^r \frac{r! \lambda^{2k}}{\lambda^{5(h_r-h_1)}} \left(\frac{1}{\lambda-1} \right)^k \\
 &\leq \sum_{h_1=1}^N \left(\sum_{m=r}^N \frac{m(m-1)\dots(m-r+1)}{r!} \frac{1}{\lambda^{5m}} r! \lambda^{2k} \left(\frac{1}{\lambda-1} \right)^k \right) \\
 &\leq \sum_{h_1=1}^N \frac{\lambda^{2k}}{(\lambda-1)^k} r! \frac{1}{(\lambda^5-1)^k} = \frac{\lambda^{2k}}{(\lambda-1)^k (\lambda^5-1)^k} r! \cdot N,
 \end{aligned}$$

and

$$\begin{aligned}
 (7a) \quad S_r &\leq \sum_{h_1=1}^N \left(\sum_{m=r}^N \frac{m(m-1)\dots(m-r+1)}{r!} \cdot \frac{1}{\lambda^{5m}} \cdot \frac{\lambda^{2k} \cdot r^k}{(\sqrt{\lambda})^{r^2}} \right) \\
 &\leq \frac{\lambda^{2k}}{(\lambda^5-1)^k} \cdot \frac{r^k}{(\sqrt{\lambda})^{r^2}} \cdot N.
 \end{aligned}$$

If $r < k^{1/2}$, we have from (6a) that

$$(6b) \quad S_r \leq \frac{\tilde{\lambda}^{2k}}{(\lambda^5 - 1)^k (\lambda - 1)^k} k \cdot k^{(k^{1/2})} \cdot N.$$

If $r \geq k^{1/2}$, then, since $k > (2^4 (\lg \lambda)^{-1})^4 = 2^{16} (\lg \lambda)^{-4}$, we have $(\tilde{\lambda})^{r^2} \geq (\tilde{\lambda})^{(k^{1/2})^2} = (\tilde{\lambda}^{k^{1/2}})^k \geq r^k$, and from (7a) we get

$$(7b) \quad S_r \leq \frac{\tilde{\lambda}^{2k}}{(\lambda^5 - 1)^k} N.$$

Hence $S = \sum_{k=1}^{\lfloor \sqrt{N} \rfloor} S_r + \sum_{k=\lfloor \sqrt{N} \rfloor + 1}^N S_r \leq \frac{\tilde{\lambda}^{2k}}{(\lambda^5 - 1)^k (\lambda - 1)^k} k \cdot k^{(k^{1/2})} \cdot N$, and this

proves Lemma 3.

REMARK. Though we shall not use it in this paper, in some cases the following variant of Lemma 3 may be useful.

LEMMA 3a. Let μ be the measure defined by (3), and $\mu_1(H)$ the measure defined as follows: if $H = \{h_1, \dots, h_m\}$, $h_1 < h_2 < \dots < h_m$, then $\mu_1(H) = \mu(H) + h_1$. Then

$$\begin{aligned} \sum_{n_1 + \dots + n_N = k} \frac{k!}{n_1! \dots n_N!} \tilde{\lambda}^{-\sum_{i=1}^N \binom{n_i + n_1 + \dots + n_N}{1 \ 2 \ \dots \ N} \Delta n_i} \\ \leq \frac{\tilde{\lambda}^{2k} k}{(\lambda^5 - 1)^k} k^{(k^{1/2})}, \end{aligned}$$

so that the number N does not appear in the right hand of the inequality.

The proof is identical to that of Lemma 3. In the present case we will have besides the factor λ^{-h_1} in the right side of (6) or (7), so that N will not appear in the last formulas (6a) and (7a).

3. Application to Hilbert and L^2 -spaces. Let $\mathbf{A} = \{T_i\}$ be a commutative normed ring. This means that $\mathbf{A}$ is a set in which a sum $T_1 + T_2 = T_2 + T_1$, a product $T_1 T_2 = T_2 T_1$, and a norm $\|T\| \geq 0$ are defined, in such a way that the following conditions are satisfied:

- a) $(T_1 + T_2) T_3 = T_1 T_3 + T_2 T_3$, b) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$,
c) $\|T_1 T_2\| \leq \|T_1\| \|T_2\|$.

We shall write $T^2 = TT$, $T^n = T^{n-1} T = T T^{n-1}$.

THEOREM 1. If $T = T_1 + \dots + T_N$ and the T_i satisfy the condition

$$(A) \quad \|T_i T_j\| \leq 2^{-(i+j)}, \quad \|T_i\| < 1,$$

then

$$(1) \quad \|T^k\| \leq 2^{3k} k^{(k/3)}, N.$$

PROOF. By the property b) of the norm we have

$$\|T^k\| \leq \sum_{\alpha_1 + \dots + \alpha_N = k} \frac{k!}{\alpha_1! \dots \alpha_N!} \|T_1^{\alpha_1} \dots T_N^{\alpha_N}\|.$$

To any term of the form $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ we assign the function

$$f(i) = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_N \\ 1 & 2 & \dots & N \end{pmatrix}.$$

In particular, if $h_1 < \dots < h_m$, to $T_{h_1} T_{h_2} \dots T_{h_m}$ corresponds the characteristic function φ_H of the set $H = \{h_1, \dots, h_m\}$.

From the hypothesis (A) we have that

$$(8) \quad \|T_{h_1} \dots T_{h_m}\| \leq \|T_{h_1} T_{h_m}\| \cdot \dots \cdot \|T_{h_2} T_{h_{m-1}}\| \dots \\ \leq 2^{-(h_m - h_2 + (h_{m-1} - h_3) + \dots)} = \left(\frac{5}{2}\right)^{-\mu(H)},$$

where $\mu(H)$ is the measure defined by (3).

On the other and, to any representation of $f(i)$ of the form

$$f(i) = \gamma_1 \varphi_{H_1}(i) + \gamma_2 \varphi_{H_2}(i) + \dots \quad (\gamma_i = \text{non-negative integers})$$

we make correspond the decomposition of $T_1^{\alpha_1} \dots T_N^{\alpha_N}$ into the factors

$$T_1^{\alpha_1} \dots T_N^{\alpha_N} = (T_{h_1'} T_{h_2'} \dots)^{\gamma_1} (T_{h_1''} T_{h_2''} \dots)^{\gamma_2} \dots,$$

where

$$(h_1', h_2', \dots) = H_1, \quad (h_1'', h_2'', \dots) = H_2, \dots$$

Using (8) we get that

$$\|T_1^{\alpha_1} \dots T_N^{\alpha_N}\| \leq \|T_{h_1'} T_{h_2'} \dots\|^{\gamma_1} \cdot \|T_{h_1''} T_{h_2''} \dots\|^{\gamma_2} \dots \\ \leq \left(\frac{5}{2}\right)^{-\mu_1(H_1) - \mu_2(H_2) - \dots} \\ \leq \left(\frac{5}{2}\right)^{-\sum \gamma_i \Delta_i} = \left(\frac{5}{2}\right)^{-\sum \left(\frac{\alpha_1 \alpha_2 \dots \alpha_N}{1 \cdot 2 \dots N}\right) \Delta^2}.$$

Applying Lemma 3 we obtain

$$\|T^k\| \leq \frac{(\sqrt[5]{2})^{2k} \cdot k \cdot k^{(k^{1/5})} \cdot N}{(\sqrt[5]{2^5} - 1)^{2k} (\sqrt[5]{2} - 1)^k} \leq \frac{2^{k/2} \cdot k \cdot k^{(k^{1/5})} \cdot N}{(\sqrt[5]{2} - 1)^k},$$

and this proves Theorem 1.

Consider now a (not necessarily complete) real Hilbert space $\mathbf{H} = \{x\}$, and an operator T defined on $\mathbf{H}$ which assigns to any element $x \in \mathbf{H}$ another element $Tx \in \mathbf{H}$. T is not required to be linear, but we assume that T satisfies the Hermitean condition :

$$(9) \quad (Tx, y) = (x, Ty)$$

for any $x, y \in \mathbf{H}$, and that there is a finite number M such that

$$\|Tx\| \leq M \cdot \|x\|, \quad \text{for all } x \in \mathbf{H}.$$

We shall denote by $\|T\|$ the smallest of such numbers M , so that

$$\|T_1 + T_2\| \leq \|T_1\| + \|T_2\| \quad \text{and} \quad \|TT_1\| \leq \|T\| \cdot \|T_1\|$$

PROPOSITION 1. *If T satisfies the condition (9) then*

$$\|T\| = (\|T^{2^m}\|)^{2^{-m}} = (\|T_k\|)^{1/k}, \quad k = 2^m,$$

for any m .

This property is well known for linear operators (Cfr. Gelfand [1]), and subsists without changes for non linear ones. Since the proof is very simple we will reproduce it here. By the Schwarz inequality

$$\begin{aligned} \|Tx\| &= \{(Tx, Tx)\}^{1/2} = \{(x, T^2x)\}^{1/2} \leq \|x\|^{1/2} \|T^2x\|^{1/2} \\ &\leq \|x\|^{1/2} \cdot \|T^2\|^{1/2} \cdot \|x\|^{1/2} = \|T^2\|^{1/2} \cdot \|x\|, \end{aligned}$$

hence $\|T\| \leq \|T^2\|^{1/2}$. On the other hand, $\|T^2\| \leq \|T\| \cdot \|T\| = \|T\|^2$, $\|T^2\|^{1/2} \leq \|T\|$. Thus $\|T\| = \|T^2\|^{1/2}$, and by iteration we obtain $\|T^{2^m}\|^{2^{-m}} = \|T\|$.

THEOREM 2. Let T be an operator satisfying condition (9). If it is possible to decompose T into a sum $T = T_1 + T_2 + \dots + T_N$ such that

$$(A) \quad \|T_i T_j\| \leq 2^{-\|i-j\|}, \quad \|T_i\| \leq 1,$$

$$(B) \quad T_i T_j = T_j T_i,$$

then $\|T\| \leq 8$.

PROOF. Since the operators $T_1, \dots, T_N$ commute they all belong to a commutative normed ring $\mathbf{A} = \{T_\alpha\}$. Applying Theorem 1 and Proposition 1, we obtain, for $k = 2^m$,

$$\|T\|^k = \|T^k\| \leq 2^{nk} \cdot k \cdot k^{(k^2)/2} \cdot N,$$

$$\|T\| \leq 8k^{1/k} \cdot k^{(1+k^2)/2} \cdot N^{1/k}.$$

Allowing k to go to infinity, we obtain $\|T\| \leq 8$.

Consider now the particular case $\mathbf{H} = L^2(R^n)$ = the class of functions $f(x)$ which satisfy

$$\|f\|_2 = \left\{ \int_{R^n} |f(t)|^2 dt \right\}^{1/2} < \infty,$$

where R^n is the n -dimensional euclidean space (or more generally a locally compact abelian group).

If $k(x) \in L^1$, that is if $k(x)$ is integrable, it defines on L^2 the linear operator

$$Tf = T_k f = (T_k f)(x) = \int_{R^n} f(x-t) k(t) dt = (f * k)(x),$$

and by a known inequality of Young (Cfr. [2], Chap. IV) we have

$$\|Tf\|_2 \leq \|k\|_1 \cdot \|f\|_2,$$

where

$$\|k\|_1 = \int_{R^n} |k(x)| dx.$$

Thus

$$\|T\| = \|T_k\| \leq \|k\|_1.$$

From Theorem 2 we obtain then at once the following:

THEOREM 2a. Let $k(x) \in L^1$. If it is possible to decompose $k(x)$ into a sum $k(x) = k_1(x) + \dots + k_N(x)$ such that

$$(A) \quad \|k_i \star k_j\|_1 \leq C \cdot 2^{-(i+j)}, \quad \|k_i\|_1 \leq C,$$

where
$$k_i \star k_j(x) = \int_{\mathbb{R}^n} k_i(x-t) k_j(t) dt,$$

then
$$\|f \star k\|_2 \leq 16C^2 \cdot \|f\|_2$$

holds for every $f \in L^2(\mathbb{R}^n)$.

4. Examples. Let $\mathbb{R}^1 = \{t\}$ be the 1-dimensional euclidean space. For each m we define on $L^2(\mathbb{R}^1)$ the operator H_m by

$$(10) \quad \begin{aligned} H_m f &= H_m f(x) = \int_{-m}^m \frac{f(x-t)}{t} dt + \int_{-m}^{-m} \frac{f(x-t)}{t} dt \\ &= \int_{|m| \leq |t| \leq m} \frac{f(x-t)}{t} dt. \end{aligned}$$

For each i we define the kernels

$$(11) \quad k_i(t) = \begin{cases} 1/t & \text{if } 2^{i-1} \leq |t| \leq 2^i \\ 0 & \text{otherwise} \end{cases}$$

$$(11a) \quad k_{-i}(t) = \begin{cases} 1/t & \text{if } 2^{-i} \leq |t| \leq 2^{-i+1} \\ 0 & \text{otherwise} \end{cases}$$

and the operators

$$(11b) \quad \begin{aligned} T_i f(x) &= f \star k_i(x) = \int_{\mathbb{R}} f(x-t) k_i(t) dt \\ &= \int_{2^{i-1} \leq |t| \leq 2^i} \frac{f(x-t)}{t} dt \end{aligned}$$

$$(11c) \quad T_{-i} f(x) = f \star k_{-i}(x) = \int_{2^{-i} \leq |t| \leq 2^{-i+1}} \frac{f(x-t)}{t} dt$$

It is clear that if $m = 2^N$, then

$$(12) \quad H_m f = H_{2^N} f = \sum_{i=-N}^N T_i f.$$

It is easy to verify the following properties of the kernels $k_i(t)$:

$$(13) \quad \int_{\mathbb{R}^1} k_i(t) dt = 0, \quad \|k_i\|_1 = 1,$$

$$(14) \quad k_i(t) = 2^{-i} k_1(2^{-i}t),$$

$$(15) \quad \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| dx \leq |t|.$$

From (14) and (15) we deduce that:

$$(15a) \quad \begin{aligned} \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| dx &= 2^{-i} \int_{\mathbb{R}^1} |k_1(2^{-i}x - 2^{-i}t) - k_1(2^{-i}x)| dx \\ &= \int_{\mathbb{R}^1} |k_1(\xi - 2^{-i}t) - k_1(\xi)| d\xi \leq 2^{-i} |t|. \end{aligned}$$

Using (13) and (15a) we obtain, for $i > j$,

$$(16) \quad \begin{aligned} \|k_i * k_j\|_1 &= \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^1} k_i(x-t) k_j(t) dt \right| dx \\ &= \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| k_j(t) dt \right| dx \\ &\leq \int_{2^{i-1} \leq |t| \leq 2^j} |k_j(t)| \left| \int_{\mathbb{R}^1} |k_i(x-t) - k_i(x)| dx \right| dt \\ &\leq 2^{-i} 2^j \|k_j\|_1 = 2^{-(i-j)}. \end{aligned}$$

COROLLARY 1. *The kernels k_i satisfy the condition (Δ) of Theorem 2a. If $H_m f$ is defined by (10), then these operators are uniformly bounded on $L^2(\mathbb{R}^1)$:*

$$(17) \quad \|H_m f\|_2 \leq Cst. \|f\|_2,$$

where the constant is independent of m .

In fact, if $m = 2^N$, it follows from (16), (12) and Theorem 2a that

$$\|H_{2^N} f\|_2 \leq 8 \|f\|_2$$

If $2^N < m < 2^{N+1}$, then $H_m f = H_{2^N} f + H'f$,
where

$$H'f = \int_{m-|t| < 2^{N+1}} + \int_{2^N < |t| < m-1} \frac{f(x-t)}{t} dt,$$

and by Young's inequality

$$\|H'f\|_2 \leq \left\{ \int_{m-|t| < 2^{N+1}} + \int_{2^N < |t| < m-1} \frac{dt}{|t|} \right\} \|f\|_2 \leq 2 \|f\|_2,$$

and this proves (17).

For any step function $f(t)$ (or for any differentiable function with compact support) the operator

$$(18) \quad Hf(x) = \int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt$$

is perfectly defined, and

$$Hf = \lim_{m \rightarrow \infty} H_m f = \sum_{n=-\infty}^{\infty} T_n f.$$

Since the step functions are dense in L^2 and by Corolary 1 the operators H_m are uniformly continuous on $L^2(\mathbb{R}^1)$, we obtain the

COROLARY 2. *The limit $\lim H_m f = Hf$ exists for any $f \in L^2(\mathbb{R}^1)$ and is a bounded operator. Hence the operator (18) admits a continuous extension to the whole space L^2 .*

The operator Hf is known as the Hilbert transform of f , or the principal value of the integral (18).

Consider now the 2-dimensional euclidean space $\mathbb{R}^2 = \{z\}$, and let us use the complex variable notation :

$$z = x + iy = |z| \cdot e^{i\theta}.$$

Let $w(\theta)$ be a function defined on $(0, 2\pi)$ which satisfies the two following conditions:

$$(19) \quad \int_0^{2\pi} w(\theta) d\theta = 0,$$

$$(20) \quad \int_0^{2\pi} |w(\theta - d(\theta)) - w(\theta)| d\theta \leq C \cdot d,$$

for any differentiable function $d(\theta)$ such that $|d(\theta)| \leq d$.

For each m we define the operator

$$(21) \quad H_m f(u) = \iint_{1/m < |z| < m} \frac{f(u-z) w(\theta)}{|z|^2} dx dy,$$

where

$$z = |z| e^{i\theta} = x + iy,$$

and for each i we define the kernels

$$(22) \quad k_i(z) = \begin{cases} \frac{w(\theta)}{|z|^2} & \text{if } 2^{i-1} \leq |z| < 2^i, \\ 0 & \text{otherwise} \end{cases} \quad z = |z| e^{i\theta}$$

$$(22a) \quad k_{-i}(z) = \begin{cases} w(\theta)/|z|^2 & \text{if } 2^{-i} < |z| < 2^{-i+1} \\ 0 & \text{otherwise} \end{cases}$$

and the operators:

$$(22b) \quad T_i f(u) = f * k_i(u) = \iint_{2^{i-1} < |z| < 2^i} \frac{f(u-z) w(\theta)}{|z|^2} dx dy$$

$$(22b) \quad T_{-i} f(u) = f * k_{-i}(u),$$

so that for $m = 2^N$ we have

$$(12a) \quad H_m f = H_{2^N} f = \sum_{i=-N}^N T_i f.$$

Using (19) and (20) it is easy to verify the following properties of the kernels k_i :

$$(13a) \quad \iint_{\mathbb{R}^2} k_i(z) dx dy = 0, \quad (z = x + iy),$$

$$(14a) \quad k_i(z) = 2^{-2i} k_1(2^{-i} z)$$

$$(15a) \quad \iint_{\mathbb{R}^2} |k_1(z-u) - k_1(z)| dz dy \leq Cst. |u|$$

Only (15a) needs a proof. Let $z_1 = z-u = \varphi_1 e^{i\theta_1}$, $z = \varphi e^{i\theta}$, and suppose that $1 \leq \varphi_1 \leq 2$ and $1 \leq \varphi \leq 2$. Then

$$\begin{aligned} |k_1(z_1) - k_1(z)| &= \left| \frac{w(\theta_1)}{\varphi_1^2} - \frac{w(\theta)}{\varphi^2} \right| \leq \\ &= \frac{|w(\theta_1) - w(\theta)|}{\varphi_1^2} + \frac{2|w(\theta)| |\varphi_1 - \varphi|}{\varphi_1^2 \varphi^2}. \end{aligned}$$

Therefore if A denotes the set of the z such that $1 \leq |z| \leq 2$, $1 \leq |z-u| \leq 2$, we shall have

$$\begin{aligned} \iint_A |k_1(z-u) - k_1(z)| dz dy &\leq \\ &\leq \left\{ \int_1^2 \varphi d\varphi \right\} \int_0^{2\pi} \frac{|w(\theta_1) - w(\theta)|}{\varphi_1^2} d\theta + \left\{ \int_1^2 \frac{2|w(t)| (\varphi_1 - \varphi)}{\varphi_1^2 \varphi^2} d\theta \right\} \\ &\leq Cst. \int_1^2 \left\{ \int_0^{2\pi} |w(\theta) - w(\theta-d(\theta))| d\theta \right\} d\varphi + Cst. (\varphi_1 - \varphi), \end{aligned}$$

where $|d(\theta)| \leq cst. |u|$, and $\varphi_1 - \varphi \leq |u|$. Hence by (20),

$$(15b) \quad \iint_A |k_1(z-u) - k_1(z)| dz dy \leq Cst. |u|.$$

Since the set where $|k_1(z-u) - k_1(z)| \neq 0$ is equal to $A + A'$ where A' is a set of measure $\leq 2|u|$, the desired inequality (15a) follows from (15b) and (14).

From (13a), (14) and (15a) we deduce, as above, the following results:

COROLARY 1a. The kernels k_i defined by (22) satisfy the condition (A) of Theorem 2a, and the operators H_m defined by (22b) are uniformly bounded on $L^2(\mathbb{R}^2)$.

COROLARY 2a. The limit $H_m f = Hf$ exists for any $f \in L^2(\mathbb{R}^2)$ and is a bounded operator which gives the principal value of the integral

$$Hf(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(u-z)w(\theta)}{|z|^2} dx dy$$

The Corolary 2a was first proved by Zygmund and Calderon [3] using the theory of Fourier transforms.

The operators H_m in the case $\mathbb{R}^n$, $n \geq 3$, are defined in a similar way and the above corolaries hold for any n .

In the following paper we shall give applications to the ergodic theorems.

REMARK. The first direct proof (without using Fourier Analysis) of Corolary 1, for the case $\mathbb{R}^1$, was given by Besicovitch and improved by Lusin [4]. Lusin's method is a very special case of our general Theorem 2 and does not apply to the n -dimensional operators H_m , if $n \geq 2$. However, Professor Zygmund and I showed that Lusin's proof still applies for the particular 2-dimensional kernel $k(z) = z^{-2} = e^{-2i\theta} / |z|^2$, that is if $w(\theta) = e^{-2i\theta}$.

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A general interpolation theorem for linear operations*

By M. COTLAR[†]

Let $\mathbf{D} = \{f(x)\}$ be the set of all step functions defined on the n -dimensional euclidean space R^n , and $T = Tf$ a linear (or semilinear) operator defined on $\mathbf{D}$, which assigns to each $f \in \mathbf{D}$ an arbitrary function $Tf(x) = (Tf)(x)$. If (1) $\|Tf\|_p \leq C_p \cdot \|f\|_p$, $p \geq 1$, and (2) $\|Tf\|_1 \leq C_1 \cdot \|f\|_1$, hold for all $f \in \mathbf{D}$, then from a theorem of M. Riesz ([2], p. 198) it follows that (3) $\|Tf\|_r \leq C_r \cdot \|f\|_r$, for every $1 < r < p$. The aim of this paper is to show that the norms $\|Tf\|_1$ in (2) can be replaced by considerably smaller values. We modify the norm $\|Tf\|_1$ in the following ways: a) by replacing the operator T by a smaller one, b) by replacing the range of integration by smaller sets, and c) by introducing small variations of the function $f(x)$. Finally we confine the functions to cubes with centers at a fixed point x .

Since we give direct proofs, the knowledge of Riesz's theorem is not assumed here, and the paper is self contained.

1. A covering lemma. We begin by proving a lemma of Vitali's type, which will be used in this and the following papers ([6]).

LEMMA 1. *Let $Q_1, Q_2, \dots, Q_k, \dots$ be a sequence of n -dimensional cubes (with sides parallel to the coordinate axes), and let l_k denote the length of the side, and x_k the center of Q_k . If*

$$a) \quad l_k \geq l_{k+1}, \quad \text{and } b) \quad x_k \in \bigcup_{i=1}^{k-1} Q_i,$$

* Received August 15, 1955.

[†] The essential parts of this and the following papers are taken from the author's Doctoral Dissertation (University of Chicago 1953).

I wish to express my gratitude to Professor A. Zygmund for his valuable and generous advice and suggestions.

then every point x of the space belongs to at most 2^n cubes Q_k ($n = \text{dimension of the space } R^n$).

PROOF. Consider first the case $n=1$. Let $x_0 \in R^1$ and let Q_i, Q_j ($i < j$) be the first two cubes of the sequence which contain the point x_0 . Since the center x_j of Q_j does not belong to Q_i and $l_j \leq l_i$, it is clear that the distance of x_0 to the set $R^1 - (Q_i \cup Q_j)$ is $> l_j/2$. Any other cube $Q_k, k > j$, has its center x_k in the set $R^1 - (Q_i \cup Q_j)$ and the length of its side is $l_k \leq l_j$, hence it cannot contain the point x_0 . This proves the lemma for $n=1$.

We shall now prove the lemma by induction on n . Assume that the lemma is true for R^{n-1} . Consider a point $x_0 \in R^n$, a hyperplane π passing through x_0 and perpendicular to one of the axes, and let E_1, E_2 be the two half-spaces determined by π . Let $Q' = Q_{i_1}, Q'' = Q_{i_2}, \dots, (i_1 < i_2 < \dots)$ be the cubes with centers $x', x'', \dots$ in the half-space E_1 , which contain x_0 , and let $P', P'', \dots$ be the intersections of $Q', Q'', \dots$ with π . The intersections $P', P'', \dots$ are $(n-1)$ -dimensional cubes with sides parallel and equal to those of $Q', Q'', \dots$, and the center $z^{(k)}$ of $P^{(k)}$ is the projection of the center $x^{(k)}$ of $Q^{(k)}$ on π . Thus, the sequence $P', P'', \dots$ satisfies condition a); it also satisfies condition b), because if it were $z_k \in P^{(i)}, i < k$, since $z^{(k)}$ is the projection of $x^{(k)}$ and the distance $(z^{(k)}, x^{(k)}) \leq l_k \leq l_i$, it would be $x^{(k)} \in Q^{(i)}$, in contradiction with the hypothesis. Thus, $P', P'', \dots$ is a sequence of $(n-1)$ dimensional cubes satisfying a) and b), and by the inductive assumption there are at most 2^{n-1} cubes $P^{(k)}$, and hence at most 2^{n-1} cubes $Q^{(k)}$ containing x_0 . Similarly there are at most 2^{n-1} cubes containing x_0 and with centers belonging to E_2 , and this proves the lemma.

The following lemma sharpens a lemma used by N. Wiener [1] and other authors.

LEMMA 2. If for each point x of a compact set $S \subset R^n$ there is given a cube $Q(x)$ with center at x , then, given $\varepsilon > 0$, it is possible to select a sequence of these cubes $Q_1 = Q(x_1), Q_2 = Q(x_2), \dots$ such that:

- 1) The measure of the set $S - (\bigcup_i Q_i) \cap S = S - \bigcup_i Q_i$ is $< \varepsilon$. That is, the points of S not covered by the Q_i belong to a set of measure $< \varepsilon$.
- 2) Every point x of the space belongs to at most 2^n cubes Q_i .
- 3) The cubes $\frac{1}{2} Q_i$ do not overlap, where $\frac{1}{2} Q$ is the cube with the same center as Q but of half the side.
- 4) There is a system of disjoint sets E_i such that $\frac{1}{2} Q_i \subset E_i \subset Q_i$, and $\bigcup_i E_i = \bigcup_i Q_i$.

PROOF. Let $l(x)$ = length of the side of the cube $Q(x)$, and $s_1 = \sup_{x \in S} l(x)$.

Since S is compact the theorem is evident if $s_1 = \infty$, thus let us assume $s_1 < \infty$. Take a point $x_1 \in S$ such that $s_1^n < (1 + \varepsilon) l_1^n$, where, $l_1 = l(x_1)$, and let Q^1 be the cube with center at x_1 and side = s_1 . Then $Q^1 \supset Q(x_1) = Q_1$ and $|Q^1 - Q_1| < \varepsilon s_1^n$, where $|Q|$ denotes the measure of Q . Consider all the points $x \in S - Q^1$, and let $s_2 = \sup_{x \in S - Q^1} l(x)$, $s_2 \leq s_1$.

Take a point $x_2 \in S - Q^1$ such that $s_2^n < (1 + 2^{-1}\varepsilon) l_2^n$, $l_2 = l(x_2)$, and let Q^2 be the cube with center at x_2 and side = $s_2 \leq s_1$. Then, $Q^2 \supset Q(x_2) = Q_2$, and $|Q^2 - Q_2| < 2^{-1}\varepsilon s_2^n, \dots$. Continuing in this way, for all $i = 1, 2, \dots$ we obtain two sequences of cubes $Q^1, Q^2, \dots$, and $Q_1, Q_2, \dots$, such that:

- a) The sequence Q^i satisfies the two conditions of Lemma 1.
- b) Q^i and Q_i have the same center x_i , $Q^i \supset Q_i$, and $|Q^i - Q_i| < 2^{-i}\varepsilon$.
- c) $x_j \notin Q^i$, if $j > i$.
- d) If $x \in S - \bigcup_{i=1}^k Q^i$, then $l(x) \leq s_k$.

We shall show that the sequence Q_i satisfies conditions 1), 2) and 3).

By Lemma 1 (a) and (b), it is clear that Q_i satisfy condition 2). Since $s_k \leq s_i$ and $x_k \notin Q^i$ if $k > i$, it is clear that the cubes $\frac{1}{2}Q^i$, and hence the cubes $\frac{1}{2}Q_i$, do not overlap. We shall prove now that $S \subset \bigcup_i Q^i$. In fact, let $x' \in S - \bigcup_i Q^i$. Then by (d) we must have $0 < l(x') \leq s_i$, and $|\frac{1}{2}Q^i| \geq 2^{-n} l(x')^n = a > 0$, for any $i = 1, 2, \dots$. Since the cubes $\frac{1}{2}Q^i$ do not overlap and S is compact, this implies that the sequence $Q^1, Q^2, \dots$ contains a finite number m of cubes. Therefore $s_{m+1} = 0$, $l(x') \leq s_{m+1} = 0$, and we arrive to a contradiction.

Thus, $S \subset \bigcup_i Q^i$, and by (b) it follows that $|S - \bigcup_i Q_i| < \sum 2^{-i}\varepsilon = \varepsilon$, so that condition 1) is satisfied.

Finally, if the sequence E_i is defined by induction as follows:

$$E_1 = Q_1 - \bigcup_{i=1}^{\infty} \frac{1}{2}Q_i, \quad E_k = (Q_k - \bigcup_{i < k} E_i) - \bigcup_{i > k} \frac{1}{2}Q_i,$$

it is easy to see that this sequence E_i satisfies condition 4).

This proves Lemma 2.

REMARK 1. From the above proof it is clear that if $l(x) =$ length of side of $Q(x)$, is a lower semicontinuous function of x , then condition 1) of lemma 2 can be replaced by the following one:

1a) S is covered by the cubes $Q_i : S \subset \bigcup_i Q_i$.

REMARK 2. By a similar argument the following generalizations can be proved:

LEMMA 1a. Let $Q_1, Q_2, \dots$ be a sequence of n -dimensional cubes and let $l_k =$ length of side of Q_k , $x_k =$ center of Q_k . If there is a number $c \geq 1$ such that:

$$a) \quad l_{k+1} \leq cl^k; \quad b) \quad x_{k+1} \in \bigcup_{i=1}^k Q_i;$$

then every point x of the space R^n belongs to at most $(2c)^n$ cubes Q_i .

LEMMA 2a. If for each point x of a compact set S there is given a cube $Q(x)$ with center at x , then it is possible to select a sequence of these cubes $Q_1 = Q(x_1), Q_2 = Q(x_2), \dots$, such that

1a) S is covered by the cubes $Q_i : S \subset \bigcup_i Q_i$.

2a) Every point x of the space belongs to at most 2^{2n} cubes Q_i .

3a) The cubes $\frac{1}{4}Q_i$ do not overlap, where $\frac{1}{4}Q$ is the cube with the same center as Q and $\frac{1}{4}$ of side.

4a) There is a system of disjoint sets E_i such that $\frac{1}{4}Q_i \subset E_i \subset Q_i$ and $\bigcup_i E_i = \bigcup_i Q_i$.

2. The associate operators P_m and P_M .

Let $R^n = \{x\}$ be the n -dimensional space, $L^r = L^r(R^n)$, $0 < r < \infty$, the set of all measurable (real) functions $f(x)$ defined on R^n and such that

$$\|f\|_r = \left(\int_{R^n} |f(x)|^r dx \right)^{1/r} < \infty,$$

and $\mathbf{D} = \{f(x)\}$ the set of all step functions defined on R^n , that is the

set of all functions of the form $f(x) = \sum_{i=1}^m \lambda_i \varphi_{Q_i}(x)$, where Q_i are n -dimensional cubes with sides parallel to the axes, and φ_Q the characteristic function of the set Q . Each function $f \in \mathbf{D}$ belongs to the spaces L^r for every r , $0 < r < \infty$, and $\mathbf{D}$, as a subset of L^r , is dense in L^r .

If $f(x)$ is a step function it takes a finite number of values $\neq 0$ and its support, that is the set $\{x: |f(x)| > 0\}$, is composed of a finite number of cubes. We shall use the following notations:

$$(1) \quad S(f) = \{x: |f(x)| > 0\} = \text{support of } f,$$

$$(1a) \quad m(f) = \min_{x \in S(f)} |f(x)|,$$

$$(1b) \quad M(f) = \max_{x \in S(f)} |f(x)|,$$

so that, if $f \neq 0$, $0 < m(f) \leq M(f)$.

For non negative functions $f(x) \geq 0$ we shall use the following notations:

$$(2) \quad f^{[\lambda]}(x) = \begin{cases} f(x) & \text{if } f(x) \geq \lambda, \lambda \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

$$(2a) \quad f_{[\lambda]}(x) = f(x) - f^{[\lambda]}(x) \quad \therefore \quad f(x) = f^{[\lambda]}(x) + f_{[\lambda]}(x),$$

$$(2b) \quad E[f > \lambda] = \text{the set of the points } x \text{ for which } f(x) \geq \lambda,$$

$$(2c) \quad |E[f \geq \lambda]| = \mu(\lambda; f) = \text{measure of } E[f \geq \lambda].$$

Thus, $\mu(\lambda) = \mu(\lambda; f)$ is the distribution function of $f(x)$.

A general tool frequently used in Analysis for the evaluation of integrals, is the following formula

$$(3) \quad \int_{R^n} |g(x)|^r dx = - \int_0^\infty \lambda^r d\mu(\lambda) = - \int_0^\infty \lambda^r d_\lambda |E[|g| \geq \lambda]|,$$

which is an easy consequence of the definition of the Lebesgue integral*.

Let T be an operator defined on $\mathbf{D}$ which assigns to any function $f \in \mathbf{D}$ an arbitrary function $Tf(x) = (Tf)(x)$, $x \in R^n$. T has not to be linear; we only require the following conditions:

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|.$$

* The proof can be found in Zygmund's book [2], Chap. IX.

T is said to be of the type p if

$$\|Tf\|_p \leq M \cdot \|f\|_p, \quad \text{for all } f \in \mathbf{D},$$

where M does not depend on f . If T is linear it is of the type p if and only if it can be extended to a bounded operator on the whole space L^p .

A theorem of M. Riesz [2]⁽²⁾ asserts that if T is simultaneously of the type p_1 and of the type p_2 , $1 \leq p_1 < p_2$, that is if

$$(4) \quad \int_{R^n} |Tf(x)|^{p_1} dx \leq O_{p_1} \cdot \int_{R^n} |f(x)|^{p_1} dx$$

and

$$(4a) \quad \int_{R^n} |Tf(x)|^{p_2} dx \leq O_{p_2} \cdot \int_{R^n} |f(x)|^{p_2} dx,$$

then T is also of the type p :

$$(4b) \quad \int_{R^n} |Tf|^p dx \leq O_p \cdot \int_{R^n} |f|^p dx,$$

for every p such that $p_1 < p < p_2$.

Here and in the following O_p will denote a constant which depends only on T and on p , but not on f .

The purpose of this paper is to show that (4b) still holds if we replace in (4) and (4a) the norms $\|Tf\|_{p_1}$ and $\|Tf\|_{p_2}$ by smaller values. In order to obtain these smaller values we proceed to modify $\|Tf\|_p$ as indicated in the introduction. In this section we will apply the first procedure as follows:

DEFINITION 1. To each operator T we assign two operators P_m and P_M defined as follows: $P_m f(x) = m(f)$ if $|Tf(x)| \geq m(f)$, and zero otherwise, $P_M f(x) = M(f)$ if $|Tf(x)| \geq M(f)$ and zero otherwise.

⁽²⁾ Riesz's theorem asserts moreover that

$$(O_p)^{(p_2-p)/p} \leq (O_{p_1})^{(p-p_1)/p_1} (O_{p_2})^{(p_2-p)/p_2}$$

Riesz considers also more general types (p, r) (see [2] page 198). However, these other aspects of the theorem will not be considered here and we hope to do it in a future paper.

We shall say that T is of the m -type p_1 if

$$(5) \quad \int_{R^n} |P_m f(x)|^{p_1} dx \leq O_{p_1} \cdot \int_{R^n} |f(x)|^{p_1} dx,$$

and of the M -type p_2 , if

$$(5a) \quad \int_{R^n} |P_M f|^{p_2} dx \leq O_{p_2} \cdot \int_{R^n} |f|^{p_2} dx.$$

Since $P_m f(x) \leq |Tf(x)|$, $P_M f(x) \leq |Tf(x)|$, the conditions (5) and (5a) are considerably weaker than conditions (4) and (4a).

PROPOSITION 1. a) T is of the m -type p , if and only if

$$(6) \quad |E[|Tf| \geq \lambda]| \leq O_p \cdot \frac{1}{\lambda^p} \int_{R^n} |f|^p dx$$

holds for every $\lambda \leq m(f)$. b) T is of M -type p , if and only if (6) holds for every $\lambda \geq M(f)$. c) T is simultaneously of the m -type p_1 and M -type p_2 , $1 \leq p_1 < p_2$, if and only if (6) holds for every $\lambda > 0$ and for every p such that $p_1 \leq p \leq p_2$.

PROOF. a) Since $\int_{R^n} |P_m f(x)|^{p_1} dx = (m(f))^{p_1} |E[|Tf| \geq m(f)]|$, it is obvious that (6) (for every $\lambda \leq m(f)$) implies (5). Conversely, assume that (5) is true, and let $\lambda \leq m(f)$. Take a cube Q , out of the support of f and of measure $|Q| = \varepsilon$, and define the function $g(x)$ as follows: $g(x) = \lambda$, if $x \in Q$, and zero otherwise. Then $\lambda = m(f+g) = m(g)$, and by hypothesis:

$$\begin{aligned} |E[|T(f+g)| \geq \lambda/2]| &\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f+g|^{p_1} dx \\ &\leq \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f|^{p_1} dx + \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \cdot \varepsilon \lambda^{p_1}, \\ |E[|Tg| \leq \lambda/2]| &\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |g|^{p_1} dx \leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \cdot \varepsilon \lambda^{p_1}. \end{aligned}$$

Therefore, since

$$\begin{aligned} |Tf| &\leq |T(f+g)| + |Tg|, \\ E[|Tf| \geq \lambda] &\subset E[|T(f+g)| \geq \lambda/2] \cup E[|Tg| \geq \lambda/2], \\ |E[|Tf| \geq \lambda]| &\leq |E[|T(f+g)| \geq \lambda/2]| + |E[|Tg| \geq \lambda/2]| \\ &\leq \frac{2^{p_1+1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f|^{p_1} dx + \varepsilon_1, \end{aligned}$$

where ε_1 is arbitrarily small. The part b) is proved in a similar way.
c) Assume that T is simultaneously of the m -type p_1 and of the M -type p_2 , $1 \leq p_1 < p_2$, and let $p_1 \leq p < p_2$. We may assume that $f \geq 0$.

Taking in account that for any λ it is true that

$$m(f^{[\lambda]}) \geq \lambda \geq M(f_{[\lambda]}),$$

and applying a), b), we shall have:

$$\begin{aligned} |E[|Tf| \geq \lambda]| &\leq |E[|T(f^{[\lambda/2]})| \geq \lambda/2]| + |E[|T(f_{[\lambda/2]})| \geq \lambda/2]| \\ &\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda/2]}|^{p_1} dx + \frac{2^{p_2} O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda/2]}|^{p_2} dx \\ &\leq \frac{2^{p_1} O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda/2]}|^{p_1} \left(\frac{|f^{[\lambda/2]}|}{\lambda/2} \right)^{p-p_1} dx \\ &\quad + \frac{2^{p_2} O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f_{[\lambda/2]}|^{p_2} \left(\frac{|f_{[\lambda/2]}|}{\lambda/2} \right)^{p-p_2} dx \\ &\leq O_p \frac{1}{\lambda^p} \int_{R^n} |f(x)|^p dx, \end{aligned}$$

and this proves Proposition 1.

DEFINITION 1a. We shall say that T satisfies the condition (p), if T satisfies condition (6) for all $\lambda > 0$.

The following proposition is a special case of an interpolation theorem due to Marcinkiewicz [3] ⁽³⁾.

PROPOSITION A. (Of Marcinkiewicz). *If T satisfies simultaneously the condition (p_1) and the condition (p_2) , then T is of the type p , for every p such that $p_1 < p < p_2$.*

PROOF. It is sufficient to consider the case $f \geq 0$. Let us denote $|E[|Tf| \geq \lambda]|$ by $\mu(\lambda)$.

Integrating (3) by parts we obtain

$$\begin{aligned} (3a) \quad \int_{R^n} |Tf|^p dx &= - \int_0^\infty \lambda^p d\mu(\lambda) = - \lim_{t \rightarrow \infty} t^p \mu(t) + \lim_{t \rightarrow \infty} p \int_0^t \lambda^{p-1} \mu(\lambda) d\lambda \\ &= p \int_0^\infty \lambda^{p-1} \mu(\lambda) d\lambda. \end{aligned}$$

Indeed, if one of the two members of (3a) is finite then both members are finite, $t^p \mu(t) \rightarrow 0$, and (3a) is true. In both members are infinite there is nothing to prove. Hence (3a) holds in any case.

From

$$|Tf(x)| \leq |T(f^{(1)})(x)| + |T(f^{(2)})(x)|$$

we deduce

$$\begin{aligned} \mu(2\lambda) &= |E[|Tf| \geq 2\lambda]| \leq |E[|T(f^{(1)})| \geq \lambda]| + |E[|T(f^{(2)})| \geq \lambda]| \\ &\leq \mu_1(\lambda) + \mu_2(\lambda). \end{aligned}$$

Since T satisfies condition (p_1) and condition (p_2) , we obtain from (3a)

$$\int_{R^n} |Tf|^p dx = p \int_0^\infty \lambda^{p-1} \mu(\lambda) d\lambda = 2^p p \int_0^\infty \lambda^{p-1} \mu(2\lambda) d\lambda.$$

⁽³⁾ Marcinkiewicz's interpolation theorem embraces also the case of the more general types (p, r) and gives a relation between the constants O_p . Marcinkiewicz stated his results without proofs. Professor Zygmund supplied the proof and indicated some interesting applications to Fourier Analysis. I used Proposition A in my Doctoral Thesis without knowing Marcinkiewicz's theorem, and I am obliged to Professor Zygmund for letting me know the results of Marcinkiewicz.

$$\begin{aligned}
 &\leq 2^p p \int_0^\infty \lambda^{p-1} \mu_1(\lambda) d\lambda + 2^p p \int_0^\infty \lambda^{p-1} \mu_2(\lambda) d\lambda \\
 &\leq 2^p p \cdot O_{p_1} \int_0^\infty \left\{ \lambda^{p-1} \frac{1}{\lambda^{p_1}} \int_{R^n} |f^{[\lambda]}(x)|^{p_1} dx \right\} d\lambda \\
 &\quad + 2^p p \cdot \int_0^\infty \left\{ \lambda^{p-1} \frac{O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f^{[\lambda]}(x)|^{p_2} dx \right\} d\lambda \\
 &= 2^p p \cdot O_{p_1} \int_{R^n} dx \int_0^\infty \left\{ \lambda^{p-1-p_1} |f^{[\lambda]}(x)|^{p_1} d\lambda \right\} \\
 &\quad + 2^p p \cdot O_{p_2} \int_{R^n} dx \int_0^\infty \left\{ \lambda^{p-1-p_2} |f^{[\lambda]}(x)|^{p_2} d\lambda \right\} \\
 &= 2^p p \cdot O_{p_1} \int_{R^n} dx \int_0^\infty \left\{ \lambda^{p-1-p_1} |f(x)|^{p_1} d\lambda \right\} \\
 &\quad + 2^p p \cdot O_{p_2} \int_{R^n} dx \int_0^\infty \left\{ \lambda^{p-1-p_2} |f(x)|^{p_2} d\lambda \right\} \\
 &= \frac{2^p p \cdot O_{p_1}}{p - p_1} \int_{R^n} |f(x)|^{p-p_1} \cdot |f(x)|^{p_1} dx \\
 &\quad + 2^p p \cdot O_{p_2} \cdot \int_{R^n} |f(x)|^{p-p_2} \cdot |f(x)|^{p_2} dx \leq O_p \int_{R^n} |f(x)|^p dx.
 \end{aligned}$$

And this proves the proposition.

From proposition A and Proposition 1 we obtain at once the following

THEOREM 1. *If T is simultaneously of the m-type p_1 and of the M-type p_2 , then T is of the type p for every p such that $p_1 < p < p_2$. In other words, the conditions (5) and (5 a) imply the condition (4 b) for every p such that $p_1 < p < p_2$.*

The theorem 1 is not true for $p \leq p_1$. For this case we have the following weaker propositions.

THEOREM 2. *The following two conditions on T are equivalent :*

- a) T is simultaneously of the m-type p_1 and of the M-type p_2 ; $1 \leq p_1 < p_2$.
- b) T satisfies the condition

$$(4c) \quad \int_S |Tf|^q dx \leq \frac{O_p}{p-\alpha} \cdot |S|^{1-\frac{\alpha}{p}} \int_{R^n} |f(x)|^p dx \Big\}^{q/p}$$

for every set S of finite measure, and α, p , such that $0 < \alpha < p_1$, $p_1 \leq p \leq p_2$.

PROOF. a) implies b): Let $y > 0$ be a fixed positive number, $\varphi_S(x)$ the characteristic function of the set S, and let us denote

$$|E|\varphi_S(x) \cdot |Tf(x)| \geq \lambda|$$

by $\mu_S(\lambda)$. Then it is clear that $\mu_S(\lambda) \leq |S| = \text{measure of } S$, and $\mu_S(\lambda) \leq |E|Tf(x)| \geq \lambda|$. Hence, using Proposition 1 and (3a), we obtain

$$\begin{aligned} \int_S |Tf(x)|^q dx &= \int_{R^n} \varphi_S(x) |Tf(x)|^q dx = \alpha \int_0^\infty \lambda^{q-1} \mu_S(\lambda) d\lambda = \alpha \int_0^y + \alpha \int_y^\infty \\ &\leq \alpha \int_0^y \lambda^{q-1} |S| d\lambda + \alpha \int_y^\infty \left\{ \lambda^{q-1} \cdot \frac{O_p}{\lambda^p} \int_{R^n} |f(x)|^p dx \right\} d\lambda \\ &\leq y^q \cdot |S| + \frac{\alpha}{p-\alpha} O_p y^{q-p} \int_{R^n} |f|^p dx. \end{aligned}$$

Taking

$$y = \left\{ \frac{1}{|S|} \int_{R^n} |f(x)|^p dx \right\}^{1/p},$$

we obtain (4c)

b) implies a): Let S be any set of finite measure contained in $E[|Tf| \geq \lambda]$, and let p be any number such that $p_1 \leq p \leq p_2$. Take $\alpha < p_1$ and apply (4c):

$$\lambda^{\alpha} |S| \leq \int_S |Tf|^{\alpha} dx \leq \frac{O_p}{p-\alpha} |S|^{1-\frac{\alpha}{p}} \left\{ \int_{R^n} |f(x)|^p dx \right\}^{\alpha/p}$$

hence

$$|S| \leq O_p \cdot \frac{1}{\lambda^p} \int_{R^n} |f(x)|^p dx.$$

Therefore T satisfies condition (p) for every p , $p_1 \leq p \leq p_2$, and this proves the theorem.

The above proof gives also the following

THEOREM 2a. *The following two conditions are equivalent: a₁) T satisfies the condition (p₀) for a fixed $p_0 \geq 1$. b₁) T satisfies (4c) for every $\alpha < p_0$. In particular the two following conditions are equivalent: a₁) T satisfies the « Kolmogorof inequality ».*

$$(6a) \quad |E[|Tf| \geq \lambda]| \leq \frac{O_1}{\lambda} \int_{R^n} |f(x)| dx, \quad \text{for every } \lambda > 0.$$

b₂) T satisfies the inequality

$$(4d) \quad \int_S |Tf(x)|^{\alpha} dx \leq \frac{O_1}{1-\alpha} |S|^{1-\alpha} \left\{ \int_{R^n} |f(x)| dx \right\}^{\alpha},$$

for every set S and every $\alpha < 1$ (or only for one value $\alpha < 1$).

From the equivalence of the conditions a₂) and b₂) it follows that if (4d) is true for one value of $\alpha < 1$ then it is true for every $\alpha < 1$.

THEOREM 2b. *If T is simultaneously of the m-type p_1 and of the M-type p_2 , $1 < p_1 < p_2$, then :*

$$(4e) \quad \int_S |Tf(x)|^p dx \leq O_{p_1, p_2} \{ |S| + \int_{R^n} |f(x)|^{p_1} (1 + \log^+ |f(x)|) dx \}.$$

In particular if T is simultaneously of the m-type 1 and of the M-type p , $p > 1$, then

$$(4f) \quad \int_S |Tf(x)| dx \leq O_p \{ |S| + \int_{R^n} |f(x)| (1 + \log^+ |f(x)|) dx \}$$

PROOF. With the same notations as in theorem 2 and Proposition A, we have that $\mu_S(2\lambda) \leq |S|$, $\mu_S(2\lambda) \leq \mu_1(\lambda) + \mu_2(\lambda)$, hence:

$$\begin{aligned} \int_S |Tf|^p dx &\leq 2^{p_1} p_1 \int_0^\infty \lambda^{p_1-1} \mu_S(2\lambda) d\lambda \leq 2^{p_1} p_1 \int_0^1 \lambda^{p_1-1} |S| d\lambda + \\ &2^{p_1} p_1 \int_1^\infty \left\{ \lambda^{p_1-1} \cdot \frac{O_{p_1}}{\lambda^{p_1}} \int_{R^n} |f^{(\lambda)}(x)|^{p_1} dx \right\} d\lambda + \\ &+ 2^{p_1} p_1 \int_1^\infty \left\{ \lambda^{p_1-1} \frac{O_{p_2}}{\lambda^{p_2}} \int_{R^n} |f^{(\lambda)}(x)|^{p_2} dx \right\} d\lambda \\ &= 2^{p_1} |S| + \int_{R^n} dx \int_1^\infty \frac{O_{p_1}}{\lambda} |f(x)|^{p_1} d\lambda + \int_{R^n} dx \int_{|f(x)|}^\infty O_{p_2} \lambda^{p_1-p_2-1} |f(x)|^{p_2} d\lambda \\ &\leq 2^{p_1} |S| + O_{p_1} \int_{R^n} |f(x)|^{p_1} \log^+ |f(x)| dx + \frac{O_{p_2}}{p_2-p_1} \int_{R^n} |f(x)|^{p_2} dx, \end{aligned}$$

and this proves the theorem.

We still observe the following property: Let T be of the M -type p_0 , $p_0 > 1$. Then T satisfies the « Kolmogoroff inequality » (5d), if and only if it satisfies the following inequality

$$(5e) \quad |E[|Tf^{(\lambda)}| > \lambda]| \leq \frac{O_1}{\lambda} \int_{R^n} |f(x)| dx, \quad \text{for all } \lambda > 0.$$

3. The main theorem. In this section we shall confine our attention to the case $p_1 = 1$ only. Without loss of generality we may assume that $p_2 = 2$.

In this section and henceforth we shall assume that the operator T satisfies the condition (2):

$$(7) \quad |E[|Tf| \leq \lambda]| \leq O_2 \cdot \frac{1}{\lambda^2} \int_{R^n} |f(x)|^2 dx$$

for every $\lambda > 0$, and every $f \in D$.

We know already that condition (7) together with condition

$$(7a) \quad \int_{R^n} |Tf(x)| dx = \|Tf\|_1 \leq O_1 \int_{R^n} |f(x)| dx = O_1 \|f\|_1,$$

implies that

$$(4b) \quad \int_{R^n} |Tf|^p dx \leq O_p \int_{R^n} |f|^p dx,$$

for every p such that $1 < p < 2$.

Now, while keeping condition (7) fixed, we shall weaken the condition (7a) by modifying the value of the norm $\|Tf\|_1$.

For this purpose we shall replace, in the first place, the range of integration R^n by the smaller set $R^n - S(f)$, where $S(f)$ is the support of f , and more generally by sets of the form $R^n - S_L(f)$, where $S_L(f)$ is a « generalized support » defined as follows. The ordinary support $S(f)$ may be defined as the set of the points x for which $|If(x)| \geq m(f)$, where $If = f$ is the identity operator. The identity operator I satisfies, obviously, the following conditions:

a) If is of the m -type 1, and of the M -type 2.

b) $|I(f+g)| < |If| + |Ig|$.

DEFINITION 2. Let $L = If$ be a fixed operator satisfying the conditions a) and b). We define the generalized support $S_L(f)$ of f as the set of the points x for which $|Lf(x)| \geq m(f)$.

Now we will introduce small variations of the function f , in the following sense: We will modify the function f only within the support S and by values not exceeding the minimum $m(f)$. More precisely:

DEFINITION 3. For each number $\Delta > 0$ and for each operator L satisfying a) and b), and each $f \in \mathbf{D}$, we define $V(f, L, \Delta) = \{\tilde{z}(x)\}$, to be the set of all the function $\tilde{z}(x)$ such that

$$(\tilde{z}_1) \quad \tilde{z}(x) = 0, \quad \text{if} \quad x \in R^n - S_L(f),$$

$$(\tilde{z}_2) \quad |\tilde{z}(x)| \leq \Delta m(f), \text{ for all } x \in R^n.$$

From the definition it is clear that if $\tilde{z} \in V(f, L, \Delta)$ and $\tilde{z}' \leq \tilde{z}$, then also $\tilde{z}'(x) \in V(f, L, \Delta)$. We will replace now f by the function $f + \tilde{z}$ which gives the minimum value to the integral. We arrive thus to the following definition.

DEFINITION 3a. For any operator L satisfying a) and b), and for any number Δ we define the « modified norm ».

$$\|Tf\|_{L, \Delta} = \inf_{\tilde{z} \in V(f, L, \Delta)} \int_{R^n - S_L(f)} |T(f - \tilde{z})(x)| dx.$$

The notation $\| \cdot \|_{L, \Delta}$ does not mean that $\| \cdot \|_{L, \Delta}$ is a norm, but only that this number was obtained by a certain modification of the norm $\| \cdot \|_1$.

Since the null function $\tilde{z} \equiv 0$ belongs to $V(f, L, \Delta)$, we have

$$\|Tf\|_{L, \Delta} \leq \int_{R^n - S_L(f)} |Tf(x)| dx \leq \|Tf\|_1.$$

In general $\|Tf\|_{L, \Delta}$ is considerably less than $\|Tf\|_1$.

PROPOSITION 2. Let T be an operator satisfying the condition (7) and the condition :

$$(7b) \quad \|Tf\|_{L, \Delta} \leq O_1 \cdot \|f\|_1,$$

for some operator L , some constant Δ , and for all $f \in \mathbf{D}$. Then T is of the m -type 1, and hence of the type p , for every p such that $1 < p < 2$.

PROOF. Let $\varepsilon < 0$. By hypothesis, there exists a function $\tilde{z}(x) \in V(f, L, \Delta)$ such that

$$(7c) \quad \int_{R^n - S_L(f)} |Tg(x)| dx \leq O_1 \int_{R^n} |f(x)| dx + \varepsilon,$$

where $g(x) = f(x) - \tilde{z}(x)$, $f(x) = g(x) + \tilde{z}(x)$.

Let us denote the set $E[|Tg| > \frac{1}{2} m(f)]$ by E_g . Then, by (7c)

$$(7d) \quad |E_g \cap (R^n - S_L(f))| \leq \frac{2 O_1}{m(f)} \int_{R^n} |f(x)| dx + \varepsilon \cdot \frac{2}{m(f)}.$$

Since the operator L is simultaneously of the m -type 1 and of the M -type 2, and therefore satisfies condition $-(p)$ for every $1 < p < 2$, we obtain

$$(7e) \quad |S_L(f)| = |E[|Lf| \geq m(f)]| \leq \frac{O_1}{m(f)} \int_{R^n} |f(x)| dx.$$

From (7d) and (7e) it follows

$$(7f) \quad |E_g| = |E_g \cap S_L(f)| + |E_g \cap (R^n - S_L(f))| \\ \leq \frac{4 O_1}{m(f)} \int_{R^n} |f(x)| dx + \frac{2\varepsilon}{m(f)}.$$

Using (7), (7e), (7f) and (7g) we have

$$(7g) \quad |E_g| = \left| E \left[|T\tilde{g}| \geq \frac{m(f)}{2} \right] \right| \leq \frac{2 O_2}{m(f)^2} \int_{R^n} |\tilde{g}(x)|^2 dx \\ \leq \frac{2 O_2 \Delta}{m(f)^2} |S_L(f)| \cdot m(f)^2 = 2 O_2 \Delta |S_L(f)| \\ \leq 2 O_2 \cdot \Delta \cdot \frac{O_1}{m(f)} \int_{R^n} |f(x)| dx = \frac{O_1'}{m(f)} \cdot \int_{R^n} |f(x)| dx.$$

Since $f = g + \tilde{g}$,

$$|E[|Tf| \geq m(f)]| \leq \left| E \left[|Tg| \geq \frac{m(f)}{2} \right] \right| + \left| E \left[|T\tilde{g}| \geq \frac{m(f)}{2} \right] \right| \\ = |E_g| + |E_{\tilde{g}}|,$$

and since $\varepsilon < 0$ is arbitrary, we obtain from (7g) and (7f),

$$|E[|Tf| \geq m(f)]| \leq \frac{O_1''}{m(f)} \int_{R^n} |f(x)| dx.$$

Hence T is of the m -type 1, and this proves the proposition.

If we define $\|T\|_1$ and $\|T\|_{L, \Delta}$ by

$$\|T\|_1 = \sup_{f \in \mathfrak{D}} \frac{\|Tf\|_1}{\|f\|_1}, \quad \|T\|_{L, \Delta} = \sup_{f \in \mathfrak{D}} \frac{\|Tf\|_{L, \Delta}}{\|f\|_1},$$

the conditions (7 a) and (7 b) may be written in the following form:

$$(7 a) \quad \|T\|_1 \leq O_1$$

$$(7 b) \quad \|T\|_{L, \Delta} \leq O_1.$$

The Proposition 2 replaces the norm $\|T\|_1$ by the smaller number $\|T\|_{L, \Delta}$. We shall now replace $\|T\|_{L, \Delta}$ by a still smaller number. For this purpose, we will give first another expression for $\|T\|_{L, \Delta}$.

Let Q be a cube with center at a fixed point x_0 and with sides parallel to the axes, $\frac{1}{2}Q$ the cube with the same center but of half side, and let E be a set composed of a finite number of cubes and such that $\frac{1}{2}Q \subset E \subset Q$. Let $\delta_0 \in V(f, L, \Delta)$ be such that

$$\|Tf\|_{L, \Delta} = \int_{R^n - S_L(f)} |T(f - \delta_0)| dx - \varepsilon.$$

Then, f and δ_0 being fixed, if Q is sufficiently large, we will have $\varphi_E(x)f(x) = \varphi_Q(x)f(x) = f(x)$, $\varphi_Q(x)\delta_0(x) = \delta_0(x)$, where φ_E is the characteristic function of the set E . Hence, for large Q , and any E such that $\frac{1}{2}Q \subset E \subset Q$, we have:

$$\inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} |T(f\varphi_E - \delta\varphi_Q)| dx \leq$$

$$\int_{R^n - S_L(f)} |T(f - \delta_0)| dx \leq \|Tf\|_{L, \Delta} + \varepsilon,$$

$$\sup_{E, \frac{1}{2}Q \subset E \subset Q} \inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} |T(f\varphi_E - \delta\varphi_Q)| dx \leq \|Tf\|_{L, \Delta} + \varepsilon.$$

On the other hand, for any $\delta \in V(f, L, \Delta)$ is also $\delta\varphi_Q \in V(f, L, \Delta)$, hence the left side of the last inequality is $\geq$ than

$$\inf_{\delta \in V(f, L, \Delta)} \int_{R^n - S_L(f)} |T(f - \delta)| dx = \|Tf\|_{L, \Delta}.$$

Thus we obtain that

$$(8a) \quad \|Tf\|_{L,\Delta} = \lim_{|Q| \rightarrow \infty} \left\{ \sup_{E, \frac{1}{2}Q \subset E \subset Q} \inf_{\substack{\varphi \in V(f, L, \Delta) \\ \int_{R^n - S_L(f)} |T(f\varphi_E - \tilde{\varepsilon}\varphi_Q)(x)| dx \leq \dots} \right\}$$

This suggests the following definition.

DEFINITION 4. For each cube $Q = Q(x_0)$ with center at x_0 we consider all the sets E such that $\frac{1}{2}Q \subset E \subset Q$, and define:

$$(9) \quad \|Tf\|_{L,\Delta,Q} = \sup_{\frac{1}{2}Q \subset E \subset Q} \inf_{\substack{\varphi \in V(f, L, \Delta) \\ \int_{R^n - S_L(f)} |T(f\varphi_E - \tilde{\varepsilon}\varphi_Q)(x)| dx \leq \dots}}$$

Then, by (8a) $\|Tf\|_{L,\Delta}$ and $\|T\|_{L,\Delta}$ may be written in the following form:

$$(8b) \quad \|Tf\|_{L,\Delta} = \lim_{|Q(x_0)| \rightarrow \infty} \|Tf\|_{L,\Delta,Q(x_0)},$$

$$(8c) \quad \|T\|_{L,\Delta} = \sup_{f \in \mathcal{D}} \lim_{|Q(x_0)| \rightarrow \infty} \frac{\|Tf\|_{L,\Delta,Q(x_0)}}{\|f\varphi_Q\|_1}$$

for any point $x_0 \in R$. We shall now replace « lim » by « inf »:

DEFINITION 4a. For each point $x_0 \in R^n$, we consider all the cubes $Q = Q(x_0)$ with center at x_0 , and define

$$(9a) \quad \|T\|_{L,\Delta,x_0} = \sup_{f \in \mathcal{D}} \inf_{Q=Q(x_0)} \frac{\|Tf\|_{L,\Delta,Q}}{\|f\varphi_Q\|_1}.$$

From (8c) it is clear that $\|T\|_{L,\Delta,x_0} \leq \|T\|_{L,\Delta}$.

THEOREM 3. Let Δ be a positive number and L an operator satisfying the above conditions a) b). If T satisfies the two conditions:

$$(7) \quad |E[|Tf| > \lambda]| \leq \frac{O_2}{\lambda^2} \int_{R^n} |f|^2 dx, \quad \text{for all } \lambda > 0,$$

and

$$(10) \quad \|T\|_{L,\Delta,x_0} \leq O_1,$$

for all $x_0 \in \mathbb{R}^n$, then T satisfies the inequality

$$(7b) \quad \|T\|_{L, 2^n \Delta} \leq O_1 = 2^n O_1.$$

Furthermore, by Proposition 2, T is of the m -type 1 and of the type p , for every p such that $1 < p < 2$.

PROOF. Let $f \in \mathbf{D}$. By assumption, for every $x_0 \in S(f)$ there is a cube $Q(x_0) = Q$, such that

$$\|Tf\|_{L, \Delta, Q} \leq O_1 \|f\varphi_Q\|_1,$$

hence, by definition of $\|Tf\|_{L, \Delta, Q}$, for any set E such that $\frac{1}{2}Q \subset E \subset Q$ there is a function $\tilde{\varepsilon}_E \in V(f, L, \Delta)$, such that

$$(10a) \quad \int_{\mathbb{R}^n - S_L(f)} |T(f\varphi_E - \tilde{\varepsilon}_E \varphi_Q)(x)| dx \leq O_1 \|f\varphi_Q\|_1, \quad Q = Q(x_0).$$

By Lemma 2, it is possible to select a sequence of these cubes $Q_1 = Q(x_1)$, $Q_2 = Q(x_2)$, ... and a sequence of sets $E_1, E_2, \dots$ with the following properties: There is a set $S_1 = S_1(f) \subset S(f)$ such that $|S(f) - S_1| < \varepsilon$ and: 1) $S_1(f) \subset \bigcup_i Q_i = \bigcup_i E_i$; 2) any point $x \in \mathbb{R}^n$ belongs to at most 2^n cubes Q_i ; 3) the sets E_i are disjoint and $\frac{1}{2}Q_i \subset E_i \subset Q_i$.

By (10a) for each i there exists a function $\tilde{\varepsilon}_i = \tilde{\varepsilon}_{E_i} \in V(f, T, \Delta)$ such that

$$(10b) \quad \int_{\mathbb{R}^n - S_L(f)} |T(f\varphi_{E_i} - \tilde{\varepsilon}_i \varphi_{Q_i})(x)| dx \leq O_1 \|f\varphi_{Q_i}\|_1.$$

Let us define

$$\tilde{\varepsilon}(x) = \sum_i \tilde{\varepsilon}_i(x) \varphi_{Q_i}(x) = \sum_i \tilde{\varepsilon}_{E_i}(x) \varphi_{Q_i}(x).$$

Since all the $\tilde{\varepsilon}_i$ vanish outside of $S_L(f)$, we have

$$\tilde{\varepsilon}(x) = 0, \quad \text{if } x \in \mathbb{R}^n - S_L(f).$$

Since any point x belongs to at most 2^n cubes Q_i , and since each

$$|\tilde{\varepsilon}_i(x)| \leq \Delta m(f),$$

we obtain that

$$|\tilde{\varepsilon}(x)| \leq 2^n \Delta, m(f), \quad \text{for all } x \in \mathbb{R}^n.$$

Thus, $\delta(x)$ satisfies the conditions (δ_1) and (δ_2) with the constant $2^n \Delta$ instead of Δ , that is $\delta(x) \in V(f, L, 2^n \Delta)$.

Since the sets E_i are disjoint, denoting $f(x) \varphi_{E_i}(x)$ by $f'_i(x)$,

$$f'(x) = \sum_i f'_i(x) \varphi_{E_i}(x),$$

$$f'(x) - \delta(x) = \sum_i (f'_i \varphi_{E_i} - \delta_i \varphi_{Q_i})(x),$$

and we obtain

$$(10c) \quad |T(f' - \delta)(x)| \leq \sum_i |T(f'_i \varphi_{E_i} - \delta_i \varphi_{Q_i})(x)|.$$

From (10c) and (10b) we deduce that

$$\begin{aligned} \int_{R^n - S_L(f)} |T(f' - \delta)(x)| dx &\leq \sum_i \int_{R^n - S_L(f)} |T(f'_i \varphi_{E_i} - \delta_i \varphi_{Q_i})(x)| dx \\ &\leq O_1 \sum_i \|f'_i \varphi_{E_i}\|_1 = O_1 \sum_i \int_{Q_i} |f(x)| dx. \end{aligned}$$

Since any point $x \in R^n$ belongs to at most 2^n cubes Q_i ,

$$\sum_i \int_{Q_i} |f(x)| dx \leq 2^n \int_{R^n} |f(x)| dx,$$

and we obtain

$$\int_{R^n - S_L(f)} |T(f' - \delta)(x)| dx \leq 2^n O_1 \int_{R^n} |f(x)| dx.$$

Since $\delta \in V(f, L, 2^n \Delta)$ and since $\varepsilon > 0$ is arbitrary, using (7) it is easy to see that the last inequality holds with f instead of f' , hence $\|Tf\|_{L, 2^n \Delta} \leq 2^n O_1 \|f\|_1$.

Hence, by proposition 2, T is of the m -type 1, and of type p , for $1 < p < 2$. This proves the theorem.

4. The modified norms $\|Tf\|_r$. The generality of the Theorem 3 makes it difficult to apply it directly. For this reason, we give in this section an intermediate theorem which is more easy to handle.

We shall call «a square-support» of $f(x)$, any cube $Q(f)$ such that $Q(f) \supset S(f)$, where $S(f)$ is the ordinary support of f .

For each square-support $Q(f)$ we define the mean value

$$(11) \quad \mu(f, Q) = \frac{1}{|Q|} \int_Q |f(x)| dx, \quad Q = Q(f).$$

DEFINITION 5. We define $W(f, Q(f), \Gamma) = \{\gamma(x)\}$, to be the class of all the functions $\gamma \in \mathbf{D}$ such that

$$\begin{aligned} (\gamma_1) \quad & \gamma(x) = 0, \quad \text{if} \quad x \in R^n - Q(f), \\ (\gamma_2) \quad & |\gamma(x)| \leq \Gamma \cdot \mu(f, Q(f)), \text{ for all } x \in R^n, \end{aligned}$$

Γ being a fixed positive number, and $Q(f)$ a square-support of f . Next, for each square-support $Q = Q(f)$, we define

$$(11a) \quad \|Tf\|_{Q, \Gamma} = \inf_{\gamma \in W(f, Q, \Gamma)} \int_{R^{n-2Q}} |T(f - \gamma)(x)| dx,$$

where $2Q$ is the cube with the same center as Q and of twice the side. Finally we define

$$(11b) \quad \|Tf\|_{\Gamma} = \sup_{Q=Q(f)} \|Tf\|_{Q, \Gamma}.$$

Let Λ be the Hardy-Littlewood maximal operator, defined by

$$(\Lambda) \quad \Lambda f(x) = \sup_{Q(x)} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)| dt \right\},$$

where the sup is taken for all the cubes $Q(x)$ containing the point x . By a known theorem of Hardy and Littlewood (the proof of this, and more general theorems, is given in the following paper [6]) the operator $L = \Lambda f$ satisfies the conditions a), b) required in Definition 2.

THEOREM 4. Let T be an operator satisfying (7). If T satisfies the inequality

$$(12) \quad \|Tf\|_{\Gamma} \leq O_1 \|f\|_1, \quad \text{for all } f \in \mathbf{D},$$

then T satisfies also the inequality

$$(12a) \quad \|Tf\|_{L, \Delta} \leq O_1' \cdot \|f\|_1,$$

with $L = 2^{2n+1} \Lambda$ and $\Delta = \frac{1}{2} \Gamma$, where Λ is the Hardy-Littlewood maximal operator. Hence T is of the m -type 1 and of the type p , for every $1 < p < 2$.

PROOF. By theorem 3 it is enough to prove that

$$\|T\|_{\epsilon_n, L, \Delta} \leq O_1,$$

for every point x_0 . Or, what is the same, it is enough to prove that given $f \in \mathbf{D}$ and $x_0 \in \mathbb{R}^n$, there is a cube $Q = Q(x_0)$, such that for every set E , $\frac{1}{2}Q \subset E \subset Q$, there exists a function $\tilde{z}_E \in V(f, L, \Delta)$ such that

$$(12b) \quad \int_{\mathbb{R}^n - S_L(f)} |T(f\varphi_E - \tilde{z}_E\varphi_Q)(x)| dx \leq O_1 \int_Q |f(x)| dx,$$

If x_0 is outside of $\mathbf{S}(f)$, (12b) will be trivially satisfied if we take Q sufficiently small and $\tilde{z}_E \equiv 0$. Therefore, let us assume that $x_0 \in \mathbf{S}(f)$.

Then we have

$$\lim_{|Q(x_0)| \rightarrow 0} \frac{1}{|Q(x_0)|} \int_{Q(x_0)} |f(x)| dx = |f(x_0)| \geq m(f).$$

For very small cubes $Q(x_0)$ we have

$$\frac{1}{|Q(x_0)|} \int_{Q(x_0)} |f(x)| dx \geq \frac{m(f)}{2}$$

while for very large cubes $Q(x_0)$ this quotient is arbitrary small. Hence there is a cube $Q = Q(x_0)$ with center at x_0 satisfying

$$(13) \quad \frac{1}{|Q|} \int_Q |f(x)| dx = \frac{m(f)}{2} \quad \therefore \quad \mu(f, Q) = \frac{m(f)}{2}.$$

We shall prove that this cube $Q = Q(x)$ is the desired one. Let E be a set such that $\frac{1}{2}Q \subset E \subset Q$ and let g be defined by $g(x) = f(x)\varphi_E(x)$. Then $Q \supset S(g) = \text{support of } g$, so that Q is a square-support for g , and

$$(14) \quad \mu(g, Q) \leq \mu(f, Q) = \frac{m(f)}{2}.$$

Hence by hypothesis $\|Tg\|_V \leq O_1 \|g\|_1$, and since $\|Tg\|_V = \sup_Q \|Tg\|_{Q,V}$, we obtain $\|Tg\|_{Q,V} \leq O_1 \|g\|_1$.

By definition of $\|Tg\|_{Q,V}$, it follows that there exists a function $\gamma(x)$ such that

$$(15) \quad \gamma(x) = 0 \quad \text{if } x \in R^n - Q,$$

$$(15a) \quad |\gamma(x)| \leq \Gamma_\mu(g, Q) \leq \frac{\Gamma}{2} m(f),$$

and

$$(16) \quad \int_{R^n - 2Q} |T(g - \gamma)(x)| dx \leq 2 O_1 \|g\|_1.$$

By (15), $\gamma(x) = \gamma(x) \varphi_Q$, and by definition $g(x) = f(x) \cdot \varphi_R(x)$, so that (16) may be written in the following form

$$(17) \quad \int_{R^n - 2Q} |T(f\varphi_R - \gamma\varphi_Q)(x)| dx \leq 2 O_1 \|f\varphi_Q\|_1.$$

If $x \in 2Q$, there is a cube Q' of measure $|Q'| = 2^{2n} |Q|$, such that $x \in Q'$ and $Q \subset Q'$. Hence, by (13)

$$\frac{1}{|Q'|} \int_{Q'} |f(x)| dx \geq \frac{1}{2^{2n} |Q|} \int_Q |f| dx = \frac{m(f)}{2^{2n+1}}.$$

This shows that $2^{2n+1} \Lambda f(x) \geq m(f)$ for every $x \in 2Q$, so that by definition of $S_L(f) = S_{2^{2n+1}\Lambda}(f)$,

$$(18) \quad R^n - S_L(f) \subset R^n - 2Q \subset R^n - Q.$$

Therefore, by (17)

$$(17a) \quad \int_{R^n - S_L(f)} |T(f\varphi_R - \gamma\varphi_Q)(x)| dx \\ \leq \int_{R^n - 2Q} |T(f\varphi_R - \gamma\varphi_Q)(x)| dx \leq 2 O_1 \|f\varphi_Q\|_1.$$

From (15), (15a) and (18) we deduce that $\gamma \in V(f, L, \frac{1}{2}\Gamma)$. Hence from (17a) we obtain (12b), and this proves the theorem.

COROLARY 1. Let T be an operator satisfying condition (7) and the condition :

$$(p) \quad \int_{R^n - 2Q} |T(f - \mu'(f, Q) \varphi_Q)(x)| dx \leq O_1 \int_{R^n} |f(x)| dx, \quad Q = Q(f),$$

for every square-support $Q = Q(f)$ and for every $f \in \mathbf{D}$, and where $\mu'(f, Q)$ is defined by

$$\mu'(f, Q) = \frac{1}{|Q|} \int_Q f(x) dx.$$

Then T is of the m -type 1 and hence satisfies the inequality

$$(4b) \quad \int_{R^n} |Tf(x)|^p dx \leq O_p \int_{R^n} |f(x)|^p dx,$$

for every p such that $1 < p < 2$.

In the following section we give an application of this Corolary.

5. Examples. The Hilbert operator

$$(19) \quad Hf(x) = \int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt = \lim_{\varepsilon \rightarrow 0} H_{\varepsilon} f(x) \\ = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon^{-1} - |t-x| > \varepsilon} \frac{f(x-t)}{t} dt,$$

is perfectly well defined, for every step function $f \in \mathbf{D}$.

We have seen in the precedent paper [5] that H is of the type 2 :

$$(20) \quad \int_{-\infty}^{\infty} |Hf(x)|^2 dx \leq O_2 \int_{-\infty}^{\infty} |f(x)|^2 dx,$$

so that H satisfies obviously condition (7).

PROPOSITION 3. The operator H satisfies the condition (p) of the precedent Corolary. The operators H_{ε} satisfy the same condition uniformly, that is with the same constant $O_1 (\leq 4)$.

PROOF. Let $Q = Q(f) = (-a, a)$ be a square support of $f(x)$, and let $\mu'(x) = \mu'(f, Q) \chi_Q(x)$. Then

$$(21) \quad \int_{-\infty}^{\infty} \mu'(t) dt = \mu'(f, Q) \cdot |Q| = \int_Q f(t) dt,$$

$$\int_{-\infty}^{\infty} (f(t) - \mu'(t)) dt = \int_{-a}^a (f(t) - \mu'(t)) dt = 0,$$

and

$$(21a) \quad \int_{-\infty}^{\infty} |\mu'(t)| dt = |\mu'(f, Q)| \cdot |Q| \leq \|f\|_1.$$

Using (21) we obtain

$$(22) \quad |H(f - \mu')(x)| = \left| \int_{-\infty}^{\infty} \frac{f(x-t) - \mu'(x-t)}{t} dt \right| = \left| \int_{-\infty}^{\infty} \frac{f(t) - \mu'(t)}{x-t} dt \right|$$

$$= \left| \int_Q \frac{f(t) - \mu'(t)}{x-t} dt \right| = \left| \int_{-a}^a \frac{f(t) - \mu'(t)}{x-t} dt - \frac{1}{x} \cdot 0 \right| =$$

$$\left| \int_{-a}^a \frac{f(t) - \mu'(t)}{x-t} dt - \int_{-a}^a \frac{f(t) - \mu'(t)}{x} dt \right| = \left| \int_{-a}^a \frac{[f(t) - \mu'(t)] t}{x(x-t)} dt \right|$$

$$\leq a \int_{-a}^a \frac{|f(t)| dt}{|x(x-t)|} + a \int_{-a}^a \frac{|\mu'(t)| dt}{|x| |x-t|}.$$

If $x \in R^1 - 2Q$, that is if $|x| \geq 2a$, and if $|t| < a$, then $|x| |x-t| \geq \frac{1}{2} |x|^2$, and by (22) and (21a) we obtain

$$(22a) \quad |H(f - \mu')(x)| \leq 4a \|f\|_1 : |x|^2$$

for every $x \in R^1 - 2Q$. Hence

$$\int_{R^1 - 2Q} |H(f - \mu')(x)| dx \leq 2 \cdot \int_{2a}^{\infty} \frac{4a \cdot \|f\|_1}{x^2} dx = 4a \cdot \|f\|_1,$$

and this proves the proposition for the operator H . The same argument applies to the H_2 .

PROPOSITION 4. The operators H and H_z are of the type p and

$$(23) \quad \int_{\mathbb{R}^n} |Hf(x)|^p dx \leq O_p \int_{\mathbb{R}^n} |f(x)|^p dx,$$

$$(23) \quad \int_{-\infty}^{\infty} |H_z f(x)|^p dx \leq O_p \int_{-\infty}^{\infty} |f|^p dx, \quad (O_p \text{ independent of } z)$$

for every $1 < p < \infty$. For $p \leq 1$ these relations should be replaced by the following weaker inequalities:

$$(23a) \quad |E[Hf] \geq \lambda| \leq \frac{O_1}{\lambda} \int_{-\infty}^{\infty} |f(x)| dx, \quad (\lambda > 0),$$

$$(23b) \quad \int_{\mathbb{R}^n} |Hf(x)|^z dx \leq \frac{O_1}{1-z} |S|^{1-z} \int_{-\infty}^{\infty} |f| dx \Big\}, \quad (z < 1),$$

$$(23c) \quad \int_{\mathbb{R}^n} |Hf(x)| dx \leq O_1 \Big\{ |S| + \int_{-\infty}^{\infty} |f(x)| (1 + \log^+ |f(x)|) dx \Big\},$$

and similarly for the operators H_z .

PROOF. From Corollary 1, Theorem 2, 2a) and 2b), and from (20) and Proposition 3, it follows that (23a), (23b), (23c) are true, and that (23) is true for $1 < p \leq 2$. If $p > 2$, and q is the conjugate number, $1/p + 1/q = 1$, then, since L^p and L^q are conjugate spaces, by a well known theorem,

$$\|Tf\|_p = \sup_g \left\{ \left| \int_{-\infty}^{\infty} T f(x) \cdot g(x) dx \right| : \|g\|_q \right\},$$

for all step functions $g \in \mathbf{D}$. By interchanging the order of integrations it is easy to check that

$$\begin{aligned} \left| \int_{-\infty}^{\infty} T f(x) \cdot g(x) dx \right| &= \left| \int_{-\infty}^{\infty} g(x) dx \int_{-\infty}^{\infty} \frac{f(x-t)}{t} dt \right| \\ &= \left| \int_{-\infty}^{\infty} T g(x) \cdot f(x) dx \right| \leq \|Tg\|_q \cdot \|f\|_p. \end{aligned}$$

Since $q < 2$, T is of the type q and we obtain

$$\|Tf\|_p \leq \frac{\|Tg\|_q \cdot \|f\|_p}{\|g\|_q} \leq \frac{O_q \cdot \|g\|_q \cdot \|f\|_p}{\|g\|_q} = O_p \cdot \|f\|_p,$$

and this proves Proposition 3.

The inequality (23) is due to M. Riesz who proved it by using the theory of analytic functions. The inequalities (23 a), (23 b) are due to Kolmogoroff.

Consider now, as in the precedent paper [5], the 2-dimensional euclidean space, $R^2 = \{z = x + iy = |z| e^{i\theta}\}$, a function $w(\theta)$ defined on $(0, 2\pi)$ which satisfies the following conditions:

$$\int_0^{2\pi} w(\theta) d\theta = 0, \quad \int_0^{2\pi} |w(\theta - d(\theta)) - w(\theta)| d\theta \leq C \cdot d,$$

if $|d(\theta)| \leq d$, and the operator Hf defined by

$$(24) \quad Hf(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(u-z) w(\theta)}{|z|^2} dx dy = \lim_{\epsilon \rightarrow 0} H_\epsilon f(u) \\ = \lim_{\epsilon \rightarrow 0} \int_{\epsilon < |z-u| < \epsilon^{-1}} \frac{f(u-z) w(\theta)}{|z|^2} dx dy.$$

By exactly the same argument as in Proposition 3 and 4 (using the inequality (15 b) of the precedent paper) we obtain:

PROPOSITION 3 a. *The operators H and H_ϵ , defined by (24), satisfy the condition (μ) uniformly, that is with a common constant O_1 .*

PROPOSITION 4 a. *The operator H defined by (24) is of the type p , that is satisfies (23), for every p such that $1 < p < \infty$; H satisfies also the conditions (23 a), (23 b) and (23 c). The corresponding operators H_ϵ fulfil the same conditions uniformly, with a common constant O_p .*

The Proposition 4 a is due to Zygmund and Calderón [4] who proved it by a different method.

The same argument applies to the case R^n , $n \geq 3$, and the Propositions 3 a and 4 a are true for any n -dimensional kernel $K(t) = \frac{w(\theta)}{|z|^n}$,

where $w(\theta)$ does not depend on $|z|$ and satisfies the above conditions. More general results will be given in the following paper on Hilbert transforms and ergodic theorems.

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Some generalizations of the Hardy-Littlewood maximal theorem*

By M. COTLAR

If $T=Tf$ is an operator of the type p , or of the m -type p (see the definition in the precedent paper), and if $M=Mf$ is another operator such that $|Mf| \leq |Tf|$ for each f , then it is obvious that Mf is also of the type p , or the m -type p . However, the situation becomes less trivial if we replace the ordinary relation $\leq$ by more general ones.

In this paper we consider two relations $|M| \prec |T|$ and $|M| << |T|$, which express that Mf «is locally in mean» $\leq |Tf|$. The maximal theorem of Hardy and Littlewood ([1], p. 244) corresponds to the case $|M| \prec |I|$ where I is the identity operator.

We also consider the case of product operators $Mh(x, y) = M_1^x(M_2^y h(x, y))$. In this case the situation remains the same if $p > 1$, but some modifications should be introduced if $p \leq 1$. Finally we give a general maximal theorem for double transformations. We give direct proofs and the Hardy-Littlewood theorem is not assumed to be known.

1. Local subordinations of operators. — Let $R^n = \{x\}$ be the n -dimensional space and $\mathbf{D} = \{f(x)\}$ a set of functions dense in all the $L^p(R^n)$ -spaces; for instance, $\mathbf{D}$ may be the set of all step functions. Let $T=Tf$ be an operator defined on $\mathbf{D}$ and such that

$$(1) \quad |T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|$$

for each $x \in R^n$, and any $f, g \in \mathbf{D}$.

As in the precedent paper [2], we shall say that T is of the type p , if

$$\|Tf\|_p^p = \int_{R^n} |Tf|^p dx \leq O_p \int_{R^n} |f|^p dx = O_p \|f\|_p^p$$

* Received August 22, 1955.

for all $f \in \mathbf{D}$. T will be said to satisfy condition (p) if

$$(2) \quad |E[|Tf| \geq \lambda]| \leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p dx,$$

for any $\lambda > 0$ and any $f \in \mathbf{D}$. Here $|E[\]|$ denotes the measure of the set $E[\]$. We have seen in the precedent paper that (2) is equivalent to

$$(3) \quad \int_S |Tf|^\alpha dx \leq O_\alpha |S|^{1-\alpha/p} \left\{ \int_{R^n} |f|^p dx \right\}^{\alpha/p}, \text{ for every } \alpha < p.$$

If $M = Mf$ is another operator defined on R^n and satisfying (1), we shall write $|M| \leq O_1 |T|$, if $|Mf(x)| \leq O_1 |Tf(x)|$ for every $f \in \mathbf{D}$ and almost all $x \in R^n$, O_1 being independent of f and x . The following proposition is trivial and we state it for the sake of completeness.

PROPOSITION 1. Let $|M| \leq O_1 |T|$. If T is of the type p , then M is also of the type p . If T satisfies condition (p), then M also satisfies this condition.

DEFINITION 1. We shall write $|M| \prec O_1 |T|$, if for each $f \in \mathbf{D}$ and for each $x \in R^n$, there is a (n -dimensional) cube $Q(x)$ with center at x (and with sides parallel to the axes), such that

$$|Mf(x)| \leq \frac{O_1}{|Q(x)|} \int_{Q(x)} |Tf(t)| dt = \frac{O_1}{|Q(x)|} \int \varphi_{Q(x)}(t) |Tf(t)| dt, \quad (A)$$

where $\varphi_Q(t)$ is the characteristic function of the set Q , and $|Q|$ the measure of Q .

The inequality (A) expresses that $Mf(x)$ is «locally subordinated» to $|Tf(x)|$, and the term «locally» refers to the point x . We consider now another interpretation of the term «locally subordinated», referring both to the point x and the function f .

DEFINITION 2. We shall write $|M| \ll O_1 |T|$, if for each $f \in \mathbf{D}$ and $x \in R^n$, there is a cube $Q(x)$, with center at x and such that

$$(B) \quad |Mf(x)| \leq \frac{O_1}{|Q(x)|} \int_{Q(x)} |T(\varphi_{Q(x)} \cdot f)(t)| dt.$$

We shall write $|M|^\alpha \ll |T|^\alpha$, if $|M_\alpha| \ll |T_\alpha|$, where $|M_\alpha f| = |Mf|^\alpha$ and $|T_\alpha f| = |Tf|^\alpha$.

If $I = If$ is the identity operator, $|M| \asymp O_1 |I|$ is equivalent to

$$|Mf(x)| \leq O_1 \cdot \sup_{Q(x)} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)| dt \right\},$$

where the sup is taken for all the cubes with center at x .

Let Λ be the Hardy-Littlewood maximal operator (see [1], p. 244), defined by

$$(\Lambda) \quad \Lambda f(x) = \sup_{Q(x)} \left\{ \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)| dt \right\}.$$

Then $|M| \asymp O_1 |T|$ is equivalent to

$$(\Lambda_1) \quad |Mf(x)| \leq O_1 \cdot \Lambda f(x).$$

PROPOSITION 2. (Of Hardy and Littlewood). *The operator Λ possesses the following properties:*

a) For any $\lambda > 0$, $E_\lambda = E[\Lambda f \geq \lambda] \subset H_\lambda$ where $H_\lambda \subset \mathbb{R}^n$ is a set satisfying

$$(4) \quad |E_\lambda| \leq |H_\lambda| \leq \frac{2^n}{\lambda} \int_{H_\lambda} |f(x)| dx \leq \frac{2^n}{\lambda} \int_{\mathbb{R}^n} |f| dx.$$

b) If $f \geq 0$ then

$$(5) \quad |E[\Lambda f \geq 2\lambda]| \leq |H_\lambda| \leq \frac{2^n}{\lambda} \int_{H_\lambda} f^{(2)}(x) dx,$$

where $f^{(2)}(x) = f(x)$ if $f(x) \geq \lambda$, and zero otherwise.

c) If $p > 1$, then $\{\Lambda f\}^p \leq \Lambda(|f|^p)$.

d) Λ satisfies condition 1, and is of the type p for every $p > 1$.

PROOF: For each point $x \in E_\lambda = E[\Lambda f \geq \lambda]$ there is a cube $Q(x)$ such that

$$\lambda \leq \Lambda f(x) \leq \frac{(1+\varepsilon)}{|Q(x)|} \int_{Q(x)} |f| dt,$$

$$\lambda |Q(x)| \leq (1+\varepsilon) \int_{Q(x)} |f(t)| dt.$$

By Lemma 2 of the precedent paper [2] we can select a sequence of

these cubes $Q_i = Q(x_i)$ such that $E_\lambda \subset \cup_i Q_i = H_\lambda$, and each point of R_n belongs to at most 2^n cubes Q_i . Therefore

$$\begin{aligned} \lambda |E_\lambda| &\leq \lambda |H_\lambda| \leq \lambda \sum_i |Q_i| \leq (1+\varepsilon) \sum_i \int_{Q_i} |f(t)| dt \\ &\leq (1+\varepsilon) 2^n \int_{\sum Q_i} |f| dt = (1+\varepsilon) 2^n \int_{H_\lambda} |f(t)| dt, \end{aligned}$$

and this proves part a).

In order to prove b), it is sufficient to observe that if $x \in E_{2\lambda}$, there is a cube $Q(x)$ such that (assuming $f \geq 0$, $f_{[\lambda]} = f - f^{[\lambda]}$),

$$2\lambda \leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} |f(t)| dt \leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f^{[\lambda]} dt + \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f^{[\lambda]} dt.$$

Since

$$\begin{aligned} \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f_{[\lambda]} dt &\leq (1+\varepsilon) \lambda, \\ (1-\varepsilon) \lambda &\leq \frac{1+\varepsilon}{|Q(x)|} \int_{Q(x)} f^{[\lambda]}(t) dt, \end{aligned}$$

and we repeat now the argument used in a).

In order to prove c), it is sufficient to observe that, by Hölder's inequality,

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f| dt &\leq \frac{1}{|Q|} \cdot |Q|^{1-1/p} \left\{ \int_Q |f|^p dt \right\}^{1/p} = \frac{1}{|Q|} \int_Q |f|^p dt \Big\}^{1/p} = \\ &\leq \Lambda(|f|^p)^{1/p}. \end{aligned}$$

From c) and a) it follows that:

$$\begin{aligned} |E[\Lambda f \geq \lambda]| &= |E[\{\Lambda f\}^p \geq \lambda^p]| \leq |E[\Lambda(|f|^p) \geq \lambda^p]| \\ &\leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p dt, \quad (p > 1). \end{aligned}$$

This shows that Λ satisfies condition (p) for every $p > 1$, and by a) Λ satisfies condition 1. Hence, by Theorem 1 of the precedent paper [2], Λ is of the type p , for every $p > 1$. This proves Proposition 1.

THEOREM 1. a) If $|M|^{\alpha} \prec O_{\alpha}|T|^{\alpha}$, and T satisfies condition (p) (respectively, if T is of the type p), and $p < \alpha$, then M also satisfies condition (p) (or is of the type p).

b) If $|M|^{\alpha} \ll |T|^{\alpha}$, and T satisfies condition (p), $p < \alpha$, then M also satisfies condition (p), and is of the type q, for any $q > p$.

PROOF. a) Let $|M|^{\alpha} \prec O_{\alpha}|T|^{\alpha}$, so that $|Mf|^{\alpha} \leq O_{\alpha} \Lambda(|Tf|^{\alpha})$. By Proposition 2, a),

$$E[|Mf| \geq \lambda] = E[|Mf|^{\alpha} \geq \lambda^{\alpha}] \subset E[O_{\alpha} \Lambda(|Tf|^{\alpha}) \geq \lambda^{\alpha}] \subset H_{\lambda},$$

and

$$|H_{\lambda}| \leq \frac{O_{\alpha}}{\lambda^{\alpha}} \int_{H_{\lambda}} |fT|^{\alpha} dt.$$

If T satisfies condition (p), then by (3),

$$|H_{\lambda}| \leq \frac{O_{\alpha}}{\lambda^{\alpha}} \int_{H_{\lambda}} |Tf|^{\alpha} dt \leq \frac{O_{\alpha}}{\lambda^{\alpha}} |H_{\lambda}|^{1-\alpha/p} \int_{R^n} |f|^p dt \lambda^{\alpha/p},$$

and we obtain $|E[|Mf| \geq \lambda]| \leq |H_{\lambda}| \leq \frac{O_{\alpha-p}}{\lambda^p} \int_{R^n} |f|^p dt.$

Hence M satisfies condition (p), and this proves the first assertion of a). On the other hand, since $|Mf|^{\alpha} \leq O_{\alpha} \Lambda(|Tf|^{\alpha})$, by Proposition 2, c)

$$\int_{R^n} |Mf|^p dt \leq O_{\alpha}^{p/\alpha} \int_{R^n} \Lambda(|Tf|^{\alpha})^{p/\alpha} dt \leq O_{\alpha-p} \int_{R^n} |Tf|^p dt.$$

Hence, if T is of the type p, we obtain

$$\int_{R^n} |Mf|^p dt \leq O_p \int_{R^n} |f|^p dt,$$

and M is also of the type p.

b) Let $|M|^{\alpha} \ll O_{\alpha}|T|^{\alpha}$, so that for each f and each $x \in R^n$,

$$|Mf(x)|^{\alpha} \leq \frac{O_{\alpha}}{|Q(x)|} \int_{Q(x)} |T(\varphi_{Q(x)} \cdot f)(t)|^{\alpha} dt.$$

If T satisfies condition (p), $p > \alpha$, then by (3)

$$\begin{aligned} |Mf(x)|^q &\leq \frac{O_a}{|Q(x)|} \cdot |Q(x)|^{1-\alpha/p} \left\{ \int_{R^n} |\varphi_{Q(x)}(t) f(t)|^p dt \right\}^{q/p} \\ &= O_a \cdot \left(\frac{1}{|Q(x)|} \right)^{q/p} \left\{ \int_{Q(x)} |f(t)|^p dt \right\}^{q/p}, \end{aligned}$$

hence

$$(4) \quad |Mf(x)|^p \leq \frac{O_p}{|Q(x)|} \int_{Q(x)} |f|^p dt \leq O_p \cdot \Lambda(|f|^p).$$

Therefore, $|M|^p \prec O_p |I|^p$, where I denotes the identity operator, and by a), M is of the type q , for every $q > p$.

On the other hand, since $E[|Mf| \geq \lambda] = E[|Mf|^p \geq \lambda^p]$, by (4) and by Proposition 2, a),

$$|E[|Mf| \geq \lambda]| < |E[O_p \Lambda(|f|^p) \geq \lambda^p]| \leq \frac{O_p}{\lambda^p} \int_{R^n} |f|^p dt.$$

Hence M satisfies condition (p), and this proves Theorem 1.

REMARK. $|M|^q \prec |T|^q$ implies also $|M|^p \prec |T|^p$, for any $p > q$. Similarly, $|M|^q \ll |T|^q$ implies $|M|^p \ll |T|^p$. In fact, if $|M|^q < |T|^q$, then $|Mf(x)|^q \leq \Lambda(|T(\varphi_{Q(x)} f)|^q)(x)$, and by Proposition 2, b), $|Mf(x)|^q \leq [\Lambda(|T(\varphi_{Q(x)} f)|^p)(x)]^{q/p}$. Hence $|Mf(x)|^p \leq \Lambda(|T(\varphi_{Q(x)} f)|^p)(x)$, that is, $|M|^p \ll |T|^p$.

COROLLARY 1. Let M, T_1, T_2, T_3 , be operators defined on $\mathbf{D}$, such that for each $f \in \mathbf{D}$ and each $x \in R^n$, there exist two cubes $Q(x)$ and $Q'(x)$, with center at x and satisfying

$$\begin{aligned} (C) \quad |Mf(x)|^q &\leq O_1 |T_1 f(x)|^q + \frac{O_2}{|Q(x)|} \int_{Q(x)} |T_2 f(x)|^q dt \\ &\quad + \frac{O_3}{|Q'(x)|} \int_{Q'(x)} |T_3(\varphi_{Q'(x)} \cdot f)(t)|^q dt, \quad (q < 0). \end{aligned}$$

Then: a) If T_1, T_2, T_3 , satisfy condition (p), $p < q$, then M also satisfies condition (p).

b) If T_2 satisfies condition (p₁), and if T_1, T_3 are of the type p₂, $q < p_1 < p_2$, then M also satisfies condition (p₁) and is of the type p₂.

PROOF. Condition (C) implies that, for each $x \in \mathbb{R}^n$, one at least of the three following inequalities must be true

$$|Mf(x)|^s \leq 3 O_1 |T_1(x)|^s, \quad (c_1)$$

$$|Mf(x)|^s \leq 3 O_2 |Q(x)|^{-1} \int_{Q(x)} |T_2 f(t)|^s dt, \quad (c_2)$$

$$|Mf(x)|^s \leq 3 O_3 |Q'(x)|^{-1} \int_{Q'(x)} |T_3(\varphi_{Q(x)} f)(t)|^s dt. \quad (c_3)$$

Therefore, $\mathbb{R}^n \subset E_1 \cup E_2 \cup E_3$, where E_i is the set of the points x satisfying (C_i) , $i=1, 2, 3$. Define $M_i f(x) = |Mf(x)|$ if $x \in E_i$, and zero otherwise, $i=1, 2, 3$. Then $|Mf(x)| \leq M_1 f(x) + M_2 f(x) + M_3 f(x)$, and $M_i^s \leq 3 O_i |T_i|^s$, $|M_2|^s \leq 3 O_2 |T_2|^s$, $|M_3|^s \leq 3 O_3 |T_3|^s$. If T_1, T_2, T_3 , satisfy condition (p), $p > \alpha$, then by Proposition 1 and Theorem 1, M_1, M_2 and M_3 satisfy condition (p), hence M satisfies the same condition. The part b) is proved in a similar way.

COROLARY 2. Let M, T_1, T_2, T_3 , be operators defined on $\mathbf{D}$, such that for each $f \in \mathbf{D}$ and for each $x \in \mathbb{R}^n$, there is a cube $Q(x)$ with center at x and satisfying

$$|Mf(x)| \leq O_1 |T_1 f(x)| + O_2 |T_2 f(x_1)| + O_3 |T_3(\varphi_{Q(x)} f)(x_1)|, \quad (D)$$

for every point $x_1 \in \frac{1}{2} Q(x)$ (= the cube with the same center as Q , and half of side).

Then :

a) If T_1, T_2, T_3 , satisfy condition (p), then M also satisfies condition (p).

b) If T_2 satisfies condition (p), and if T_1, T_3 are of the type p_1 , $p < p_1$, then M also satisfies condition (p) and is of the type p_1 .

PROOF. Let $\alpha < p$. Since (D) holds for each point $x_1 \in \frac{1}{2} Q(x)$, then, raising to the α -power and integrating over $\frac{1}{2} Q(x)$ we obtain :

$$\begin{aligned} |Mf(x)|^s &= \frac{2^n}{|Q(x)|} \int_{Q(x)/2} |Mf(x)|^s dx_1 \leq 2^n O_1 \int_{Q(x)/2} |T_1 f(x)|^s dx_1 + \\ &+ \frac{1}{|Q(x)|} \int_{Q(x)/2} |T_2 f(x_1)|^s dx_1 + \frac{1}{|Q(x)|} \int_{Q(x)/2} |T_3(\varphi_{Q(x)} f)(x_1)|^s dx_1. \end{aligned}$$

Hence by Corolary 1 we obtain Corolary 2.

$R^{n+m} = R^n \times R^m = \{ (x, y) \}$, $L^p(R^{n+m})$ will denote the set of the functions $h(x, y)$ defined on R^{n+m} and such that

$$\|h\|_p^p = \int_{R^{n+m}} |h(x, y)|^p dx dy = \int_{R^m} dy \int_{R^n} |h(x, y)|^p dx < \infty,$$

and $L^p(R^n)$, $L^p(R^m)$, the corresponding spaces for R^n and R^m , respectively.

Let $T = Tf = Tf(x)$ (respectively $S = Sg(y)$) be an operator defined on the set $\mathbf{D}(R^n)$ (respectively, $\mathbf{D}(R^m)$) of all step functions $f(x)$ (respectively $g(y)$) of R^n (of R^m). We assume that T and S are linear operators.

Let $\mathbf{D}(R^n \times R^m) = \mathbf{D}(R^n) \times \mathbf{D}(R^m) = \{ h(x, y) \}$ be the set of all step functions of R^{n+m} of the form

$$h(x, y) = \lambda_1 f_1(x) g_1(y) + \dots + \lambda_k f_k(x) g_k(y),$$

where $f_i \in \mathbf{D}(R^n)$, $g_i \in \mathbf{D}(R^m)$, and the λ_i are constants, $i = 1, 2, \dots, k$. The sets $\mathbf{D}(R^n)$, $\mathbf{D}(R^m)$, $\mathbf{D}(R^{n+m})$ are dense in all the L^p spaces. We define the following product operators:

$$\widetilde{Th}(x, y) = (\widetilde{Th})(x, y) = [Th(\cdot, y)](x) = \sum \lambda_i g_i(y) \cdot [Tf_i](x).$$

$$\widetilde{Sh}(x, y) = (\widetilde{Sh})(x, y) = [Sh(x, \cdot)](y) = \sum \lambda_i f_i(x) \cdot [T_i g](y).$$

$$\widetilde{S} \widetilde{Th}(x, y) = \widetilde{T} \widetilde{Sh}(x, y) = [\widetilde{T} \widetilde{Sh}](x, y) = \sum \lambda_i [Tf_i](x) \cdot [Sg_i](y).$$

PROPOSITION 4. a) If T is of the type p , or if T satisfies condition (p), then $\widetilde{T}$ is also of the type (p), or $\widetilde{T}$ satisfies condition (p), respectively.

b) If T and S are of the type p , then $\widetilde{T} \widetilde{S}$ is also of the type p .

c) If T satisfies condition (p), and S is of the type p , then $\widetilde{T} \widetilde{S}$ satisfies condition (p).

d) If T and S satisfy condition (p), then for every $\alpha < p$, and $E \subset R^{n+m}$,

$$\int_E |\widetilde{T} \widetilde{Sh}|^p dx dy \leq O_{\alpha, p} \cdot |E|^{1-\alpha/p} \cdot \int_{R^n} \int_{R^m} |\widetilde{Sh}(x, y)|^p dx dy \}^{1/p}.$$

PROOF. *a)* and *b)* are immediate consequences of Fubini's theorem. For instance,

$$\begin{aligned} \iint_{R^{n+m}} |\widetilde{TSh}|^p dx dy &= \int_{R^m} dy \int_{R^n} |\widetilde{T(Sh(x, y))}|^p dx \leq \\ &= O_p \int_{R^m} dy \int_{R^n} |\widetilde{Sh(x, y)}|^p dx = O_p \int_{R^n} dx \int_{R^m} |\widetilde{Sh(x, y)}|^p dy \\ &\leq O_{p'} \int_{R^n} dx \int_{R^m} |h(x, y)|^p dy = O_{p'} \iint_{R^{n+m}} |h(x, y)|^p dx dy. \end{aligned}$$

In order to prove *c)* and *d)*, it is sufficient to observe that condition *(p)* is equivalent to (3), and that (if $E \subset R^n \times R^m$),

$$\begin{aligned} \int_E |\widetilde{TSh}|^q dx dy &= \int_{R^m} dy \int_{R^n} \varphi_E |\widetilde{TSh}|^q dx \leq \\ &= O_p \int_{R^m} dy \cdot |E_y|^{1-\alpha/p} \int_{R^n} |\widetilde{Sh(x, y)}|^p dx \{^{q/p} \leq \\ &= O_p \int_{R^m} |E_y| dy \{^{\frac{p-q}{p}} \int_{R^m} dy \int_{R^n} |\widetilde{Sh(x, y)}|^p dx \{^{q/p} \\ &= O_p \cdot |E|^{1-\alpha/p} \int_{R^n \times R^m} |\widetilde{Sh(x, y)}|^p dx \{^{q/p}, \end{aligned}$$

where E_α denotes the intersection of the set E with the line $y = \alpha$.

Let us consider now instead of the space R^1 the unit circumference C^1 , and instead of R^n the torus $C^n = C^1 \times C^1 \times \dots \times C^1$. In this case C^n has finite measure and all the functions $f(x)$ defined on C^n will be assumed to be periodic functions. All the precedent results apply to C^n . If T is defined on $\mathbf{D}(C^n)$ and satisfies condition *(p)*, then, since C^n is of finite measure

$$\int_{C^n} |Tf|^\alpha dt \leq O_\alpha \int_{C^n} |f(t)|^p dt \{^{\alpha/p},$$

for $\alpha < p$.

Moreover, from the last inequality of the proof of Proposition 4, and from Theorem 2 of the precedent paper, it follows immediately that ;

PROPOSITION 4 a). Let T and S be defined on C^n and C^m , respectively. If T and S satisfy condition (p) and condition (p₁), $p < p_1$, then for every $\alpha < p$,

$$\left| \iint_{C^n \times C^m} |\widetilde{T} \widetilde{S} h(t, s)|^{\alpha} ds dt \leq O_{\alpha} \left\{ \iint_{C^{n+m}} |h|^p (1 + \log^+ |h|) ds dt \right\}^{\alpha/p} + O_{\alpha}'.$$

DEFINITION 3. We shall write $\widetilde{M} <_y <_x \widetilde{T}$, if for each $h(x, y) \in D(R^n \times R^m)$ and for each point $(x, y) \in R^{n+m}$, there is a cube $Q(x) \subset R^n$ with center at x , and a cube $Q(y) \subset R^m$ with center at y , such that

$$|\widetilde{M} h(x, y)| \leq \frac{O_1}{|Q(x)| |Q(y)|} \iint_{Q(x) \times Q(y)} |\widetilde{T} h(t, s)| dt ds.$$

We shall write $\widetilde{M} < <_x <_y \widetilde{T}$, if

$$|\widetilde{M} h(x, y)| \leq \frac{O_1}{|Q(x)| |Q(y)|} \iint_{Q(x) \times Q(y)} |\widetilde{T}(\varphi_{Q(x)} \cdot h)(t, s)| dt ds,$$

where $\varphi_{Q(x)}(t, s)$ is the characteristic function of the cube $Q(x) \subset R^n \subset R^{n+m}$. We shall write $\widetilde{M} < <_x < <_y \widetilde{T}$, if

$$|\widetilde{M} h(x, y)| \leq \frac{O_1}{|Q(x)| |Q(y)|} \iint_{Q(x) \times Q(y)} |\widetilde{T}(\varphi_{Q(x)} \cdot \varphi_{Q(y)} \cdot h)(t, s)| dt ds.$$

THEOREM 2. a) If $\widetilde{M} <_x <_y \widetilde{S} \widetilde{T}$, and if T and S are of the type $p > 1$, then $\widetilde{M}$ is also of the type p .

b) If $\widetilde{M} <_x < <_y \widetilde{T} \widetilde{S}$, if S is of the type p , and if T is of the type p_1 , $p_1 > p > 1$, then $\widetilde{M}$ is of the type p_1 .

c) If $\widetilde{M} < <_x < <_y \widetilde{T} \widetilde{S}$, and if T and S are of the type p , then M is of the type p_1 , for every $p_1 > p$.

PROOF. a) Let Λ^x be the Hardy-Littlewood maximal operator acting on the space $R^n = \{x\}$, and Λ^y the same operator acting on $R^m = \{y\}$. By the assumption, $\tilde{M} <_x <_y \tilde{T} \tilde{S}$, hence $|\tilde{M} h| \leq |\tilde{\Lambda}^x \tilde{\Lambda}^y \tilde{T} h|$. By Proposition 2, c and Proposition 4a)

$$\iint_{R^{n+m}} |\tilde{M} h|^p ds dt \leq O_p \iint_{R^{n+m}} |\tilde{T} \tilde{S} h|^p ds dt \leq O_{p'} \iint_{R^{n+m}} |h(s, t)|^p ds dt,$$

so that $\tilde{M}$ is of the type p .

b) By assumption, for each $h(x, y)$ and each $(x, y) \in R^{n+m}$, there is a cube $Q(x) \subset R^n$, and a cube $Q(y) \subset R^m$, such that

$$\begin{aligned} |\tilde{M} h(x, y)| &\leq \frac{O_1}{|Q(x)| |Q(y)|} \iint_{Q(x) \times Q(y)} |\tilde{T} \tilde{S}(\varphi_{Q(y)} h)(t, s)| dt ds \\ &\leq \frac{O_1}{|Q(x)|} \int_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\tilde{S} \tilde{T}(\varphi_{Q(y)} h)(t, s)| ds \right\} dt \\ &\leq \frac{O_1}{|Q(x)|} \int_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\tilde{S} \tilde{T}(\varphi_{Q(y)} h)(t, s)|^p ds \right\}^{1/p} dt. \end{aligned}$$

Taking in account that S satisfies condition (p), by Proposition 4, a) and by Proposition 2, b)

$$\begin{aligned} |\tilde{M} h(x, y)| &\leq \\ &O_1 |Q(x)|^{-1} \int_{Q(x)} \left\{ |Q(y)|^{-1} \int_{Q(y)} (\varphi_{Q(y)}(s) |\tilde{T} h(t, s)|^p ds \right\}^{1/p} dt \\ &\leq O_1 \tilde{\Lambda}^x(\{ \tilde{\Lambda}^y(|Th|^p) \}^{1/p}) \leq O_1 \{ \tilde{\Lambda}^x(\tilde{\Lambda}^y(|Th|^p)) \}^{1/p}, \end{aligned}$$

Hence, by Proposition 2, c), by Proposition 4, a), and by the assumption,

$$\begin{aligned} \iint_{R^{n+m}} |\tilde{M} h|^p dx dy &\leq O_1^{p'} \iint_{R^{n+m}} \{ \tilde{\Lambda}^x(\tilde{\Lambda}^y(|Th|^p)) \}^{p/p'} dx dy \\ &\leq O_{p'} \iint_{R^{n+m}} \{ \Lambda^y(|Th|^p) \}^{p/p'} dx dy \end{aligned}$$

$$\leq O_{p_1} \iint_{R^{n+m}} |Th|^{p_1} dx dy \leq O_{p_1} \iint_{R^{n+m}} |h|^{p_1} dx dy,$$

and this shows that $\tilde{M}$ is of the type p_1 , for every $p_1 > p$,

c) As in b), we shall have

$$\begin{aligned} |\tilde{M}h(x, y)| &\leq \frac{1}{|Q(x)|} \int_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\tilde{T}(\varphi_{Q(x)}h)(t, s)|^p ds \right\}^{1/p} dt \\ &\leq \left\{ \frac{1}{|Q(x)|} \int_{Q(y)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\tilde{T}(\varphi_{Q(x)}h)(t, s)|^p ds \right\} dt \right\}^{1/p} \\ &\leq \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} \left\{ \frac{1}{|Q(x)|} \int_{R^n} |\tilde{T}(\varphi_{Q(x)}h)(t, s)|^p dt \right\} ds \right\}^{1/p} \\ &\leq \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} \left\{ \frac{1}{|Q(x)|} \int_{R^n} \varphi_{Q(x)} |h(t, s)|^p dt \right\} ds \right\}^{1/p} \\ &\leq \tilde{\Lambda}^y \tilde{\Lambda}^x (|h|^p)^{1/p} \end{aligned}$$

Hence

$$\begin{aligned} \iint_{R^{n+m}} |\tilde{M}h|^{p_1} dx dy &\leq \iint_{R^{n+m}} \tilde{\Lambda}^y \tilde{\Lambda}^x (|h|^p)^{1/p} dx dy \\ &\leq \iint_{R^{n+m}} |h(x, y)|^{p_1} dx dy, \end{aligned}$$

and this proves Theorem 2.

THEOREM 2a. Let T and S be operators acting on the spaces C^n and C^m , respectively, and satisfying conditions (1) and (p), $p > 1$.

a) If $\alpha < 1$ and $|\tilde{M}|^\alpha <_{x < y} |\tilde{S}\tilde{T}|^\alpha$, then

$$(8) \quad \iint_{C^{n+m}} |Mh|^\beta dx dy \leq O_\beta \left\{ \iint_{C^{n+m}} |h| (1 + \log^+ |h|) \right\}^{\frac{\beta}{\beta-1}} + O_\beta',$$

for every β such that $\alpha < \beta < 1$.

b) If $\alpha < 1$, and $|\tilde{M}|^\alpha <_{x < y} |\tilde{T}\tilde{S}|^\alpha$, then (8) is true for any β , $\alpha < \beta < 1$.

c) If $\alpha < 1$, and $|\tilde{M}|^\alpha < \alpha < \alpha_y |\tilde{T}\tilde{S}|^\alpha$, then (8) is true for $\alpha < \beta < 1$.

PROOF. a) If $|\tilde{M}|^\alpha < \alpha < \alpha_y |\tilde{T}\tilde{S}|^\alpha$, then $|\tilde{M}h|^\alpha \leq \tilde{A}^\alpha \tilde{A}^{\alpha_y} (|\tilde{T}\tilde{S}h|^\alpha)$.
By Proposition 2 and Proposition 4

$$\iint_{C^{n+m}} |\tilde{M}h|^\alpha dt ds \leq O_\beta \iint_{C^{n+m}} \tilde{A}^\alpha \tilde{A}^{\alpha_y} (|\tilde{T}\tilde{S}h|^\alpha)^{\beta/\alpha} dt ds \leq O_\beta \iint_{C^{n+m}} |\tilde{T}\tilde{S}h|^\alpha dt ds,$$

and by Proposition 4 a) it follows (8).

b) We have

$$\begin{aligned} |\tilde{M}h(x, y)|^\alpha &\leq \frac{1}{|Q(x)|} \int_{Q(x)} \left\{ \frac{1}{|Q(y)|} \int_{Q(y)} |\tilde{T}\tilde{S}(\varphi_{Q(y)}h)|^\alpha ds \right\} dt \\ &\leq \frac{O_\alpha}{|Q(x)|} \int_{Q(x)} \frac{1}{|Q(y)|} \left\{ \int_{C^m} \varphi_{Q(y)}(s) |\tilde{T}h| ds \right\}^\alpha dt \\ &\leq O_\alpha \tilde{A}^\alpha (|\tilde{A}^{\alpha_y}(\tilde{T}h)|)^\alpha, \\ \iint_{C^{n+m}} |\tilde{M}h(x, y)|^\alpha dx dy &\leq O_\beta \iint_{C^{n+m}} [\tilde{A}^\alpha (|\tilde{A}^{\alpha_y}(\tilde{T}h)|)^\alpha]^{\beta/\alpha} dz dy \\ &\leq O_\beta \iint_{C^{n+m}} |\tilde{A}^{\alpha_y}(\tilde{T}h)|^\alpha dx dy \leq O_\beta' \iint_{C^{n+m}} |\tilde{T}h|^\alpha dx dy, \end{aligned}$$

and by Proposition 4, and by the Theorem 2 of the precedent paper, we obtain (8).

The part c) of the theorem is proved in a similar way.

4. A maximal theorem for product operators. — Let T_ε ($\varepsilon > 0$) be a set of linear operators defined on $\mathbf{D}(R^n)$ (or on $\mathbf{D}(C^n)$) and satisfying the following condition:

For each $\varepsilon > 0$, for each $f \in \mathbf{D}(R^n)$, and for each $x_0 \in R^n$, there is a cube $Q = Q(x_0, \varepsilon)$ with center at x_0 and such that for every point $x_1 \in \frac{1}{2}Q$, it is true that

$$(E) \quad |T_\varepsilon f(x_0)| \leq O_1 \{ |Tf(x_1)| + |T(\varphi_{Q(x_0)}f)(x_1)| + N_1 f(x_0) \},$$

where O_1 does not depend on ε and f . The operator N_1 has not to be linear but only monotonic: $|f| \leq |f'|$ implies $N_1(|f|) \leq N_1(|f'|)$.

Similarly let $S_\varepsilon (\varepsilon > 0)$ be a set of operator on $\mathbf{D}(R^m)$ (or $\mathbf{D}(C^m)$) satisfying

$$(E') \quad |S_\varepsilon g(y_0)| \leq O_1 \{ |Sg(y_1)| + |S(\varphi_{Q(y_0)} g)(y_1)| + N_2 g(y_0) \}$$

The operators $T_\varepsilon, S_\varepsilon$ define on $\mathbf{D}(R^n \times R^m)$ a set of operators

$$\widetilde{T}_\varepsilon \widetilde{S}_\varepsilon = \widetilde{T}_\varepsilon \widetilde{S}_\varepsilon.$$

Let $\widetilde{M}h(x, y)$ be the maximal operator defined on $\mathbf{D}(R^n \times R^m)$ by

$$\widetilde{M}h(x, y) = \sup_{\varepsilon > 0, \varepsilon' > 0} |\widetilde{T}_\varepsilon \widetilde{S}_{\varepsilon'}(x, y)|. \quad (M)$$

THEOREM 3. If the operators $T_\varepsilon, S_\varepsilon, N_\varepsilon, S$, and T , commute, then :

a) If the operators N_1, N_2, S, T are of the type p , for every $p > 1$, then $\widetilde{M}$ is also of the type p , for every $p > 1$.

b) If the above operators are defined on the space C^n and C^m , and if N_1, N_2, S and T satisfy the condition (1) and the condition (p), $p > 1$, then

$$(F) \quad \iint_{C^{n+m}} |\widetilde{M}h(x, y)|^2 dx dy \leq O_\alpha \left\{ \iint_{C^{n+m}} |h| (1 + \log^+ |h|)^2 + O_\alpha \right\}$$

for every $\alpha < 1$.

PROOF. Let us define $M^x f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|$, $M^y g(y) = \sup_{\varepsilon > 0} |S_\varepsilon g(y)|$.

Then by the assumption, for every $(x_0, y_0) \in R^{n+m}$, and for any $(x_1, y_1) \in {}^{1/2}Q(x_0) \times {}^{1/2}Q(y_0)$ (we may assume that $O_1 = 1$):

$$\begin{aligned} |\widetilde{T}_\varepsilon \widetilde{T}_{\varepsilon'} h(x_0, y_0)| &\leq \\ &|\widetilde{T} \widetilde{S}_\varepsilon h(x_1, y_0)| + |\widetilde{T}(\varphi_{Q(x_0)} \cdot \widetilde{S}_{\varepsilon'} h)(x_1, y_0)| + \widetilde{N}_1 \widetilde{S}_{\varepsilon'} h(x_0, y_0) \\ &\leq |\widetilde{T} S_\varepsilon h(x_1, y_0)| + |\widetilde{T}(\varphi_{Q(x_0)} \cdot \widetilde{S}_{\varepsilon'} h)(x_1, y_0)| + \widetilde{N}_1 \widetilde{M}^y h(x_0, y_0), \\ |\widetilde{T} \widetilde{S}_\varepsilon h(x_1, y_0)| &= |\widetilde{S}_\varepsilon \widetilde{T} h(x_1, y_0)| \leq |\widetilde{S} \widetilde{T} h(x_1, y_1)| + |\widetilde{S}(\varphi_{Q(y_0)} \widetilde{T} h)(x_1, y_1)| \\ &\quad + \widetilde{N}_2 \widetilde{T} h(x_1, y_0), \\ |\widetilde{T}(\varphi_{Q(x_0)} \widetilde{S}_{\varepsilon'} h)(x_1, y_0)| &\leq |\widetilde{S}(\varphi_{Q(x_0)} \widetilde{T} h)(x_1, y_1)| + |\widetilde{S} \widetilde{T}(\varphi_{Q(x_0)} \varphi_{Q(y_0)} h)(x_1, y_1)| \\ &\quad + N_2 T(\varphi_{Q(x_0)} h)(x_1, y_1). \end{aligned}$$

Taking ε, τ_1 such that $|\tilde{M}h(x_0, y_0)| < 2|\tilde{T}\tilde{S}h(x_0, y_0)|$, and raising to the α -power, we obtain:

$$\begin{aligned} |\tilde{M}h(x_0, y_0)|^\alpha &\leq O_\alpha \{ |\tilde{T}\tilde{S}h(x_1, y_1)|^\alpha + |\tilde{T}\tilde{S}(\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha + \\ &|\tilde{T}(\varphi_{Q(x_0)}\tilde{S}h)(x_1, y_1)|^\alpha + |\tilde{T}\tilde{S}(\varphi_{Q(x_0)}\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha \\ &+ |\tilde{N}_1\tilde{M}^v h(x_0, y_0)|^\alpha + |\tilde{N}_2\tilde{T}h(x_1, y_0)|^\alpha \\ &+ |\tilde{N}_2\tilde{T}(\varphi_{Q(x_0)}h)(x_1, y_0)|^\alpha \}, \end{aligned}$$

for every $\alpha \leq 1$, and $(x_1, y_1) \in {}^{1/2}Q(x_0) \times {}^{1/2}Q(y_0)$.

Integrating over the set ${}^{1/2}Q(x_0) \times {}^{1/2}Q(y_0)$, in x_1, y_1 , we shall obtain that

$$\tilde{M} \leq \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 + \dots + \tilde{M}_7,$$

where

$$\begin{aligned} |\tilde{M}_1|^\alpha &<_{x < y} |\tilde{T}\tilde{S}|^\alpha, & |\tilde{M}_2|^\alpha &<_{x < y} |\tilde{T}\tilde{S}|^\alpha, \\ |\tilde{M}_3|^\alpha &<_{x < y} |\tilde{T}\tilde{S}|^\alpha, & |\tilde{M}_4|^\alpha &<_{x < y} |\tilde{T}\tilde{S}|^\alpha, \\ |\tilde{M}_5|^\alpha &\leq |\tilde{N}\tilde{M}^v|^\alpha, & |\tilde{M}_6|^\alpha &<_x |\tilde{N}_2\tilde{T}|^\alpha, \\ |\tilde{M}_7| &<_{x < y} |\tilde{N}_2\tilde{T}|^\alpha. \end{aligned}$$

Hence by theorem 2, and by theorem 2a, this proves theorem 3.

From Theorem 3, and Proposition 3a, and Corolary 3a, we obtain the

THEOREM 3a. Let $K_1(z), K_2(u)$ be two kernels of the form (5a), defined in the space R^n and R^m (or C^n and C^m) respectively, and

$$\begin{aligned} \tilde{H}_{\varepsilon, \eta} h(z, w) &= \iint_{\substack{|z-u| < \varepsilon \\ |w-v| < \eta}} K_1(z-u) K_2(u-v) h(u, v) du dv \\ &= \tilde{H}_\varepsilon \tilde{H}_\eta h(z, w), \end{aligned}$$

the double Hilbert transform on $R^n \times R^m$, and

$$\tilde{M}h(z, w) = \sup_{\varepsilon, \eta > 0} |\tilde{H}_{\varepsilon, \eta} h(z, w)|$$

Taking ε, τ , such that $|\widetilde{M}h(x_0, y_0)| < 2|\widetilde{T}\widetilde{S}h(x_0, y_0)|$, and raising to the α -power, we obtain:

$$\begin{aligned} |\widetilde{M}h(x_0, y_0)|^\alpha &\leq O_\alpha \{ |\widetilde{T}\widetilde{S}h(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha + \\ &|\widetilde{T}(\varphi_{Q(x_0)}\widetilde{S}h)(x_1, y_1)|^\alpha + |\widetilde{T}\widetilde{S}(\varphi_{Q(x_0)}\varphi_{Q(y_0)}h)(x_1, y_1)|^\alpha \\ &+ |\widetilde{N}_1\widetilde{M}^\nu h(x_0, y_0)|^\alpha + |\widetilde{N}_2\widetilde{T}h(x_1, y_0)|^\alpha \\ &+ |\widetilde{N}_2\widetilde{T}(\varphi_{Q(x_0)}h)(x_1, y_0)|^\alpha \}, \end{aligned}$$

for every $\alpha \leq 1$, and $(x_1, y_1) \in {}^{1/2}Q(x_0) \times {}^{1/2}Q(y_0)$.

Integrating over the set ${}^{1/2}Q(x_0) \times {}^{1/2}Q(y_0)$, in x_1, y_1 , we shall obtain that

$$\widetilde{M} \leq \widetilde{M}_1 + \widetilde{M}_2 + \widetilde{M}_3 + \dots + \widetilde{M}_7,$$

where

$$\begin{aligned} |\widetilde{M}_1|^\alpha &<_x <_y |\widetilde{T}\widetilde{S}|^\alpha, & |\widetilde{M}_2|^\alpha &<_x <_y |\widetilde{T}\widetilde{S}|^\alpha, \\ |\widetilde{M}_3|^\alpha &<_x <_y |\widetilde{T}\widetilde{S}|^\alpha, & |\widetilde{M}_4|^\alpha &<_x <_y |\widetilde{T}\widetilde{S}|^\alpha, \\ |\widetilde{M}_5|^\alpha &\leq |\widetilde{N}\widetilde{M}^\nu|^\alpha, & |\widetilde{M}_6|^\alpha &<_x |\widetilde{N}_2\widetilde{T}|^\alpha, \\ |\widetilde{M}_7|^\alpha &<_x |\widetilde{N}_2\widetilde{T}|^\alpha. \end{aligned}$$

Hence by theorem 2, and by theorem 2a, this proves theorem 3.

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THEOREM 3a. Let $K_1(z), K_2(u)$ be two kernels of the form (5a), defined in the space R^n and R^m (or C^n and C^m) respectively, and

$$\begin{aligned} \widetilde{H}_{\varepsilon, \tau} h(z, w) &= \iint_{\substack{|z-u| \leq \varepsilon \\ |w-v| \leq \tau}} K_1(z-u) K_2(u-v) h(u, v) du dv \\ &= \widetilde{H}_\varepsilon \widetilde{H}_\tau h(z, w), \end{aligned}$$

the double Hilbert transform on $R^n \times R^m$, and

$$\widetilde{M}h(z, w) = \sup_{\varepsilon, \tau > 0} |\widetilde{H}_{\varepsilon, \tau} h(z, w)|$$

Then $\tilde{M}$ is of the type p for every $p > 1$, and $\tilde{M}$ satisfies the inequality (F).

(Of course it is necessary not to confuse the double Hilbert transform $\tilde{H}_{n,m}$ with the simple $(n+m)$ -dimensional Hilbert transform).

In the case $n = m = 1$, $K_1(x) = K_2(x) = x^{-1}$, theorem 3a was proved by S. Sokolowsky and Zygmund ([4], [5]) using the theory of double trigonometrical series and analytic functions.

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ALGUNAS GENERALIZACIONES DEL TEOREMA MAXIMAL DE HARDY y LITTLEWOOD

Por MISCHA COTLAR

Si $T = Tf$ es un operador del tipo p , o del m -tipo p (ver las definiciones en el trabajo que precede), y si $M = Mf$ es otro operador tal que $|Mf(P)| \leq |Tf(P)|$ para toda f , entonces es evidente que Mf es también del mismo tipo que Tf . Sin embargo el asunto ya no es tan inmediato si la relación ordinaria $\leq$ se reemplaza por otras más generales.

En este trabajo consideramos dos relaciones: $M \prec T$ y $M \ll T$ que generalizan la relación ordinaria $\leq$, y que expresan que M es $\prec T$ «localmente y en media». El teorema maximal de Hardy-Littlewood (ver [1], p. 244, en la bibliografía que precede) corresponde al caso $T = I =$ operador identidad y $M \prec T$.

Estudiamos también el caso de operadores productos $M_x M_y f f(x, y)$. En este caso la situación permanece la misma si $p > 1$, pero es necesario introducir modificaciones si $p = 1$. Finalmente damos un teorema maximal general para transformadas dobles.

Como damos demostraciones directas, para la lectura no es necesario el conocimiento del teorema de Hardy-Littlewood.

A unified theory of Hilbert transforms and ergodic theorems

BY MISCHA COTLAR

The analogy between the theory of Hilbert transforms and the ergodic theorems, and in particular the differentiation theory, has been repeatedly stressed by several authors, particularly by Lusin and Zygmund. However, the two theories have been treated by entirely different methods, the proofs for the Hilbert transforms being considerably more complex than those of the ergodic theorems. This is due to the fact that the ergodic theory deals with positive operators, while the Hilbert transforms are non-positive operators.

The aim of this paper is to give a general theory which contains the theory of Hilbert transforms and ergodic theorems as special cases. Let $R^n = \{x\}$ be the n -dimensional euclidean space, $K(x)$ an integrable function, and let $K_i(x) = 2^{-ni} K(2^{-i}x)$, $i = \pm 1, \pm 2, \pm 3, \dots$. If $\Omega = \{P\}$ is an abstract space with a measure μ and a n -dimensional continuous group $\{\sigma_x\}$, $x \in R^n$, of measure preserving transformations, we define for each $m = 1, 2, \dots$ the operator

$$H_m f(P) = \sum_{i=-m}^m \int_{R^n} f(\sigma_x P) K_i(x) dx.$$

We prove that, under certain assumptions on the kernel $K(x)$, $H_m f(P)$ converges pointwise to a limit $Hf(P)$, almost everywhere, for every $f \in L^p(\Omega, \mu)$, and every $p \geq 1$. If $p > 1$, $H_m f$ also converges in the p^{th} mean to Hf , and the maximal operator $Mf = \sup |H_m f|$ is bounded in $L^p(\Omega, \mu)$.

If $\Omega = R^n$, $\sigma_x t = x + t$, and $K(x) = \omega(x)/|x|^n$ for $1 \leq |x| \leq 2$ and zero otherwise ($\omega(x)$ does not depend on $|x|$), then $Hf = \lim H_m f$ is the n -dimensional Hilbert transform and we obtain the classical results of Lusin-Riesz-Kolmogoroff [1] concerning the ordinary Hilbert transform, as well as the recent ones of Zygmund-Calderón [2]

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concerning the generalized n -dimensional Hilbert transforms. If Ω is a general measure space, and $K(x) = -1$ if $|x| < 1$, $K(x) = +1$ if $1 \leq |x| \leq 2$, then the $H_m f$ reduce to the ergodic operators, and we obtain the ergodic theorems of von Neumann, Birkhoff and Wiener [3]. Besides, we give similar theorems for the double operators $H_{mn} f(P, Q) = H_m H_n f(P, Q)$, which contain as special cases the n -parametric ergodic theorems due to Zygmund and Dunford [4] and the theorems concerning the double Hilbert transform due to Zygmund and Sokolowsky [5]. Moreover, these theorems extend, in particular, the Zygmund-Sokolowsky results for n -dimensional kernels.

In the proof of the above general theorems we use only measure-theoretical methods. In fact, our proofs are based on three general theorems concerning operators in L^p -spaces, which have been given in the three precedent papers ([6], [7], [8]).

If we are interested in the case $\Omega = R^n$ only, the theory simplifies considerably, and as was shown in the examples of the just mentioned papers, yields in particular a new and simplified treatment of the theory of Hilbert transforms.

On the other hand, we show that using the Tauberian theorems for topological groups, the precedent theory can be extended for kernels $K(x)$ defined in locally compact abelian groups. Among other things this permits the unification of the «discrete» and «continuous» theories, such as the M. Riesz theory of discrete Hilbert transforms of sequences and the ordinary Hilbert transforms of functions. As another byproduct we obtain also an extension of the ergodic theorems to the general groups due to Calderón [9]. From our point of view, the Hilbert transforms and ergodic theorems may be considered as special cases of the tauberian theorems.

Of course, in the present paper are extended to the general operators H_m only the first most basic facts from the theory of Hilbert transforms or ergodic theory. It seems to us that the extension of the more profound features of these theories could open an interesting field of research. (Cfr. § 6, C and D).

1. Introduction. In this section we will recall the fundamental facts about Hilbert transforms and ergodic theorems, and explain the main idea of the paper.

A) The fundamental theorem of the theory of integration asserts that the indefinite integral of an integrable function is differentiable almost everywhere.

If $f(x)$ is integrable on every finite interval of the line $(-\infty, \infty)$, and if for each $\varepsilon > 0$ we define the transform $D_\varepsilon f$ by

$$D_\varepsilon f = D_\varepsilon f(x) = [D_\varepsilon f](x) = \frac{1}{\varepsilon} \int_0^\varepsilon f(x+t) dt = \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(t) dt,$$

then this theorem asserts that

$$\lim_{\varepsilon \rightarrow 0} D_\varepsilon f(x) = f(x), \text{ for almost all } x.$$

If $p > 1$, then it is also true that

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} |D_\varepsilon f(x) - f(x)|^p dx = 0, \text{ for } f \in L^p.$$

Moreover, if Λf is defined by

$$\Lambda f(x) = \sup_{\varepsilon > 0} |D_\varepsilon f(x)|,$$

then the Hardy-Littlewood maximal theorem asserts that (cfr. [8])

$$\int_{-\infty}^{\infty} |\Lambda f(x)|^p dx \leq O_p \int_{-\infty}^{\infty} |f(x)|^p dx \quad (p > 1).$$

Here O_p depends on p alone. The precedent inequality is not true if $p=1$. In this case, we have instead the following inequalities:

$$\int_{-A}^A |\Lambda f(x)|^q dx \leq O_q \cdot A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} |f(x)| dx \right\}^\alpha, \text{ if } 0 < \alpha < 1,$$

$$\int_{-A}^A |\Lambda f(x)| dx \leq O_A \left\{ \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) dx \right\} + O_A'.$$

It was observed by N. Wiener [3] that the above theorems concerning the operators D_ε are particular cases of theorems for a wider class of operators arising in the ergodic theory. In fact, if for each real number t , we define the translation $\sigma_t x = x + t$, then each σ_t is a measure-preserving transformation on $R^1 = (-\infty, \infty)$, in the sense that for each measurable set $E \subset R^1$, $\sigma_t E$ is also measurable and of the same measure as E , and $D_\varepsilon f$ may be written in the following form

$$D_\varepsilon f(x) = \frac{1}{\varepsilon} \int_0^\varepsilon f(\tau_t x) dt.$$

The ergodic theory considers, more generally, an abstract measure space $\Omega = \{P\}$ and a group of measure-perserving transformations $\{\tau_t P\}$, $(-\infty < t < \infty)$ of Ω . $D_N f$ is now defined similarly by

$$D_N f(P) = \frac{1}{N} \int_0^N f(\tau_t P) dt.$$

The ergodic theorem asserts that the above properties of the operators $D_\varepsilon f(x)$ hold for these more general operators $D_N f(P)$. The pointwise and mean convergence theorems correspond here, respectively, to the Birkhoff and von Neumann ergodic theorems, and the Hardy-Littlewood's maximal theorem to the Wiener dominated theorem. The only unimportant difference is that in the case of the ergodic theorems the limit is taken as $N \rightarrow \infty$, instead of $\varepsilon \rightarrow 0$.

It is important to note that the operators D_ε , or D_N , are positive operators, in the sense that $D_N f(x) \geq 0$ almost everywhere if $f(x) \geq 0$ for almost all x . This property of D_N is essential in the current proofs of the ergodic theorems, — those proofs do not apply to non-positive operators.

B) An important case of non positive operators, for which the above theorems still hold is exhibited by the theory of Hilbert transforms. The (ordinary) Hilbert transform Hf of the function $f(x)$, $(-\infty < x < \infty)$, is defined by

$$(I) \quad Hf(x) = \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt.$$

Hf is understood as the limit, as $\varepsilon \rightarrow 0$, of $H_\varepsilon f$, where $H_\varepsilon f$ is defined for each $\varepsilon > 0$ by

$$(Ia) \quad H_\varepsilon f(x) = \int_{|t-x|>\varepsilon} \frac{f(t)}{x-t} dt = \int_{-\infty}^{x-\varepsilon} \frac{f(t)}{x-t} dt + \int_{x+\varepsilon}^{\infty} \frac{f(t)}{x-t} dt.$$

Lusin, Privaloff and Plessner proved the pointwise convergence of $H_\varepsilon f$: for every $f \in L^p$, $p \geq 1$, the limit

$$(II) \quad \lim_{\varepsilon \rightarrow 0} H_\varepsilon f(x) = Hf(x)$$

exists for almost all x . The limit function $Hf(x)$ is then taken as the definition of the singular integral (I).

While the function $Hf(x)$ exists, it may not be integrable. M. Riesz has shown that if $f \in L^p$, and $p > 1$ then also $Hf \in L^p$ and $H_\epsilon f$ converges to Hf in the p^{th} -mean, i.e.

$$(III) \quad \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} |Hf(x) - H_\epsilon f(x)|^p dx = 0, \quad \text{for } f \in L^p, p > 1.$$

Moreover, the following inequality of M. Riesz

$$(IV) \quad \int_{-\infty}^{\infty} |Hf(x)|^p dx \leq O_p \int_{-\infty}^{\infty} |f(x)|^p dx, \quad (p > 1),$$

holds for any $f \in L^p$, where O_p depends on p alone.

Kolmogoroff proved that for $f \in L^1$ the limit operator satisfies the inequality

$$(V) \quad \int_{-A}^A |Hf(x)|^q dx \leq O_\alpha A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} |f(x)| dx \right\}^\alpha, \quad \text{if } 0 < \alpha < 1.$$

This inequality was completed by Zygmund as follows:

$$(VI) \quad \int_{-A}^A |Hf(x)| dx \leq O_A \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) dx + O'_A.$$

Finally Zygmund [1] proved the maximal theorems for H_ϵ : if $Mf(x)$ is defined by

$$(VII) \quad Mf(x) = MHf(x) = \sup_{0 < t < 1} |H_t f(x)|,$$

then

$$(VII a) \quad \int_{-\infty}^{\infty} |Mf(x)|^p dx \leq O_p \int_{-\infty}^{\infty} |f(x)|^p dx \quad (\text{if } p > 1),$$

$$(VII b) \quad \int_{-A}^A |Mf|^q dx \leq O_\alpha A^{1-\alpha} \left\{ \int_{-\infty}^{\infty} |f| dx \right\}^\alpha \quad (\text{if } 0 < \alpha < 1),$$

$$(VII c) \quad \int_{-A}^A |Mf| dx \leq O_A \int_{-\infty}^{\infty} |f| (1 + \log^+ |f|) dx + O'_A.$$

Thus, though the operators H_ϵ are not positive they possess the same properties as the positive operators D_ϵ or D_N . According to

Lusin, the pointwise convergence of the operator D_ϵ expresses the « fundamental differential property of the first order » of integrable functions, which is due to the « smallness » of $|f(x+\epsilon) - f(x)|$ as $\epsilon \rightarrow 0$. Lusin considers the corresponding property (II) of the operators H_ϵ as the « fundamental differential property of the second order » of integrable functions, which holds no longer because of the smallness of $|f(x+\epsilon) - f(x)|$, but because of the cancellation of the positive and negative values of $f(x+\epsilon) - f(x)$.

C) Though the properties of the Hilbert transforms expresses fundamental facts of real variable theory the proofs of Lusin, Riesz and Kolmogoroff were based entirely on the theory of analytic functions. For this reason, and also with a view to further generalizations, Lusin proposed the problem of giving a direct proof by real variables methods. This problem was solved by Besicovitch in two important papers [10] and [11]. The method of Besicovitch were perfected by Titchmarsh [12], and Pollard [13] extended Besicovitch's results to Hilbert-Stieljes transforms.

The methods of Besicovitch and Titchmarsh were developed by Zygmund and Calderón [2] who extended the above theorems of the Hilbert transform to a very wide class of n -dimensional transforms which arise in potential theory (cfr. Mijlin [14]). Their generalization is the following. Letting $K(x) = x^{-1}$, (I) may be written as a convolution with the kernel $K(x)$:

$$Hf(x) = f * K(x) = \int_{-\infty}^{\infty} K(x-t)f(t) dt.$$

We note that the kernel $K(x) = x^{-1}$ is not integrable near the points $x = 0$, $x = \infty$, and that $n = 1$ is the only number such that x^{-n} is simultaneously not integrable at 0 and ∞ . Moreover as we have observed already, the property $K(x) + K(-x) = 0$ is essential for the existence of $Hf(x)$ because of the interference of the positive and negative values.

Hence if we want to consider in the plane $R^2 = \{z\}$, $z = \{x, y\}$, an analogue of the kernel $K(x) = x^{-1}$, it is natural to take the kernels $K(z)$ of the form

$$K(z) = \frac{\omega(\theta)}{|z|^2}, \quad z = |z|e^{i\theta},$$

where the function $\omega(\theta)$ does not depend on $|z|$ and satisfies

$$\int_0^{2\pi} \omega(\theta) d\theta = 0.$$

The last condition corresponds to the property $K(x) + K(-x) = 0$ of $K(x) = x^{-1}$, and $|z|^{-2}$ in the denominator of $K(z)$ corresponds to the fact that $n = 2$ is the only number such that $|z|^{-n}$ is not integrable, both at $z = 0$ and $z = \infty$, over the plane R^2 . To each kernel $K(z)$, $z \in R^2$, with the above properties, corresponds a two dimensional Hilbert transform

$$(I') \quad Hf(z) = f * K(z) = \int_{R^2} K(z-u) f(u) du,$$

defined as the limit of

$$(I' a') \quad H_\varepsilon f(z) = \int_{|u| > \varepsilon} K(z-u) f(u) du,$$

as $\varepsilon \rightarrow 0$.

Similar definitions are made in the n -dimensional case, $n \geq 2$.

Under very general assumptions on the function $\omega(\theta)$, Zygmund and Calderón prove that the theorems (II)-(IV) and (VII a) hold for the n -dimensional Hilbert transforms (I'), thus obtaining a generalization of the classical theory to n -dimensional kernels $K(z)$.

D) In [2] Zygmund and Calderón do not consider the double transforms, or the cartesian product of transforms of type (I'). In the classical case, that is the 1-dimensional case of the kernel $K(x) = x^{-1}$, the double Hilbert transform is defined by

$$\begin{aligned} (I b) \quad Hf(x, y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(s, t) ds dt}{(x-s)(y-t)} \\ &= \int_{R^2} K(x-s) K(y-t) f(s, t) ds dt \\ &= \lim_{\varepsilon, \eta \rightarrow 0} H_{\varepsilon, \eta} f(x, y), \quad (K(x) = x^{-1}), \end{aligned}$$

$$(I b') \quad H_{\varepsilon, \eta} f(x, y) = \int_{\substack{|x-s| > \varepsilon \\ |y-t| > \eta}} K(x-s) K(y-t) f(s, t) ds dt.$$

Of course it is important not to confuse the double transform (Ib) of 1-dimensional kernels $K(x) = x^{-1}$, with the simple transforms (I') of 2-dimensional kernels $K(z)$. There is only one double transform (Ib) but an infinity of 2-dimensional transforms (I'). While in

(Ia') the integral is taken over the exterior of a sphere $|u-z| > z$, in (Ib') it is taken over the exterior of a « cross » $|x-s| > z$, $|y-t| > r$. More generally, we may consider double transforms of n -dimensional kernels, for instance :

$$(1c) \quad Hf(z, \omega) = \lim_{\substack{r \rightarrow \infty \\ r \neq 0}} \iint_{R^2 \times R^2} K_1(z-u) K_2(\omega-v) f(u, v) du dv = \lim_{r \rightarrow \infty} H_{r,r} f(z, \omega),$$

where $K_1(u)$, $K_2(v)$, are 2-dimensional kernels of the type $K(z) = \omega(\theta)/|z|^2$. The double Hilbert transform (Ib) of 1-dimensional kernels has been studied by Sokolowski [5] and Zygmund [5a] from the point of view of the theory of the double Fourier series and analytic functions of two variables. Consequently their proofs do not apply to the general double transform (Ic) of n -dimensional kernels.

The extension of the Sokolowski-Zygmund results to the n -dimensional kernels and to much more general double transforms, will be given below (cfr. [8]).

Finally, M. Riesz has studied the discrete Hilbert transform of sequences, instead of functions, and in the paper of Zygmund and Calderón [2] these results of Riesz are extended to n -dimensional kernels. From the general theorems given below will follow that Sokolowski-Zygmund results extend to the double discrete Hilbert transforms of n -dimensional kernels.

E) Thus, there is a complete analogy between the fundamental theorems of Hilbert transforms and ergodic theory, though the proofs in the case of Hilbert transforms are considerably more complex.

The aim of the paper is to unify and to generalize both theories. As the first step towards unification, we will try to find a common method of proof for both theories, based on general properties of operators in the L^p spaces.

For this purpose let us observe first that in the case of the Hilbert transforms as well in the ergodic theory, the proof of the convergence theorem preceded that of the maximal theorem. Pitt and Yosida showed, however that the proofs of the ergodic theorems become considerably simplified if, contrary to historical development, the maximal theorem is proved first. It is natural to expect that a similar inversion of order should be advantageous also in the case of Hilbert transforms.

It should be observed next that, in both theories, the convergence theorem is immediate for certain good functions forming a set $\mathbf{D} = \{f\}$ which is dense in all L^p -spaces. In the case of Hilbert transforms such « good » function are, for instance all the step functions or all the differentiable functions vanishing outside of compact sets. Thus for any $f \in \mathbf{D}$ the operators Hf and Mf are well defined.

Since the operator Mf is defined on $\mathbf{D}$, and since the functions $f \in \mathbf{D}$ are more easy to handle, our first idea is to reduce the problem to the proof of the properties of the maximal operator M and for the functions $f \in \mathbf{D}$ only. That is, we first prove (cfr. § 2 below) that the truth of the properties (II) – (VIIc) for all functions $f \in L^p$, and all p , depends only upon the verification of the maximal theorem (VII), — and that for the functions $f \in \mathbf{D}$ only.

Hence the proof of the ergodic theorems and those of Hilbert transform are reduced to the following general problem :

PROBLEM A. *Let $\{H_m\}$ be a set of linear bounded operators on L^p ($p > 1$) such that $H_m f(x) \rightarrow Hf(x)$, almost everywhere, for any $f \in \mathbf{D}$, where $\mathbf{D}$ is dense in all the L^p -spaces. Under what conditions does the maximal operator $Mf(x) = \sup |H_m f(x)|$ satisfy on $\mathbf{D}$ the inequalities (VII) ?*

In the precedent paper [8] we have shown that under certain general assumptions (which are satisfied in the case of Hilbert transforms or ergodic theory) Mf will satisfy the inequalities (VII) if the limit operator $Hf(x) = \lim H_m f(x)$ satisfies these inequalities on $\mathbf{D}$.

Thus, problem A is reduced to the following general problem.

PROBLEM B. *Let $\{H_m\}$ be as in problem A. Under what conditions are the operators H_m and $H = \lim H_m$ uniformly bounded on $\mathbf{D}$, considered as a subset of L^p ?*

In the precedent paper [6] we have given a solution to problem B for the case $p = 2$. Next in [7], assuming that H is bounded on L^2 , we gave general conditions for the boundedness of H on L^p for $p \neq 2$.

The conditions given in the precedent papers apply to, both, Hilbert transform and ergodic theorems. In this way we obtain a common method of proof for both theories. Moreover this method simplifies considerably the proofs for the case of Hilbert transforms, specially because of the reduction to the set $\mathbf{D}$ where most precautions concerning the existence of limit are unnecessary.

F) The above method applies not only to Hilbert transforms and ergodic theorems, but also to more general operators, thus enabling us to unify and generalize both theories as follows.

We first observe that, since the operator

$$D_1 f(P) = \int_0^1 f(\sigma_t P) dt$$

is bounded in all the L^p spaces, the ergodic theorems will not alter if instead of the operators $D_n f(P)$ we consider the operators

$$D_m' f(P) = D_{2^m} f(P) - D_1 f(P).$$

The operator $D_m' f$ (which will be denoted now by $D_m f$) may be written

$$\begin{aligned} D_m' f(P) &= \frac{1}{2^m} \int_0^{2^m} f(\sigma_t P) dt - \int_0^1 f(\sigma_t P) dt \\ &= \int_{-\infty}^{\infty} f(\sigma_t P) d^{(m)}(t) dt \end{aligned}$$

where $d^{(m)}(t) = 2^{-m} \varphi_m(t) - \varphi_0(t)$, $\varphi_m(t)$ being the characteristic function of the interval $(0, 2^m)$.

If Ω were the euclidean line $\Omega = R^1 = \{x\}$ and $\sigma_t P = \sigma_t x = x + t$, then (D) could be written as a convolution

$$D_m' f(x) = \int_{-\infty}^{\infty} f(x+t) d^{(m)}(t) dt = [f \star d^{(m)}](x).$$

Thus the ergodic operators D_n may be considered as generalized convolutions of functions f of an abstract space with kernels $d^{(m)}(x)$ of the euclidean space R^n .

On the other hand, in the case of the Hilbert transforms, the corresponding operators H_m are convolution of f with the kernels $K^{(m)}$ such that: $K^{(m)}(x) = \omega(x)/|x|^2$ if $|x| < m$ and zero otherwise where $\omega(x)$ does not depend on $|x|$.

Thus, both sets of operators, D_m and H_m are given by (generalized) convolutions with corresponding sequences of kernels $d^{(m)}(x)$ and $K^{(m)}(x)$.

Moreover: the kernels $K^{(m)}$, as well as the kernels $d^{(m)}$, are generated by dilatations of a fixed kernel, in the following sense. Let $K_i(x)$

be defined by $K_i(x) = \omega(x)/|x|^n$ if $2^i \leq |x| \leq 2^{i+1}$, and zero otherwise (n-dimension of R^n). Since $w(x)$ does not depend on $|x|$, the kernel $K(x) = K_0(x) = \omega(x)/|x|^n$ if $1 \leq |x| \leq 2$, and zero otherwise, generates the kernels K_i by the formula

$$K_i(x) = 2^{-ni} K(2^{-i}x),$$

and

$$K^{(m)}(x) = \sum_{i=-m}^m K_i(x).$$

Similarly, the kernels $d^{(m)}$ are generated by dilatations of the fixed kernel $d(t) = -1$ if $|t| < 1$, $d(t) = 1$ if $1 \leq |t| \leq 2$, and zero otherwise, in the sense that $d^{(m)} = \sum_{i=0}^m d_i$, where $d_i(t) = (2^{-i})^n d(2^{-i}t)$.

Hence the operators D_m and H_m may be written as a sum of operators E_i and T_i ,

$$(VIII) \quad H_m = \sum_{i=-m}^m T_i, \quad T_i f = f * K_i,$$

$$(VII a) \quad D_m = \sum_{i=0}^m E_i, \quad E_i f = f * d_i,$$

where the operators E_i, T_i are generated by the fixed operators E_0 and T_0 .

This suggests to consider general operators H_m on Ω generated by dilatations of an arbitrary kernel $K(x)$. We actually prove that the properties (II)–(VII c) of the operators D_m and H_m admit a generalization to such general operators H_m .

In other words, we will show that the fact that a sequence H_m of operators is generated, in the above sense, by a fixed operator, implies that the operators $H_m f(x)$ converge pointwise and in the mean, and are uniformly bounded on all the L^p spaces, $p > 1$. This particularity of the operators H_m is due to the following two fundamental properties of the generated kernels K_i :

a) If K satisfies a general Lipshitzien condition of the form $\|K^b - K\|_1 \leq O_1 \cdot |b|$, where $K^b(t) = K(t+b)$, then the generated kernels K_i satisfy the condition $\|K_i^b - K_i\|_1 \leq O_1 2^{-i} |b|$, and this implies as we will see that $\|T_i T_j f\|_2 \leq O_1 \cdot 2^{-|i-j|} \cdot \|f\|_2$. Thus $H_m = T_2 + \dots + T_m$ is a decomposition of H_m in «almost ortogonal» operators, in the sense that for large values of $|i-j|$ the operators T_i and T_j are almost ortogonal. In the precedent paper [6] we just

proved that operators which admit such «almost orthogonal» decompositions have their norms bounded by fixed constants. Thus, the operators H_m generated by a «smooth» kernel are uniformly bounded on L^2 .

b) The fact that K_i are generated by dilatations of a fixed kernel $K(t)$ which compact support implies that $x = 0$ is the only point such that $\hat{K}_i(x) = 0$ for all $i = \pm 1, 2, \dots$, where $\hat{K}$ is the Fourier transform of K . Therefore, by a tauberian theorem of Wiener-Beurling-Kaplansky, any function $f \in L^1(R^1)$ such that $\int_{-\infty}^{\infty} f(t) dt = 0$, can be approximated in L^1 by linear combinations of translations of the kernel K_i . From this fact it is easy to deduce (see §4 below) that the operators $H_m f(P)$ converge pointwise for all f of a set $\mathbf{D}$ dense in all the space $L^p(\Omega)$. Thus, the profound reason of the pointwise convergence of the operators H_m lies in this property of the Fourier transforms $\hat{K}_i$, which in its turn is due to the generating property of the K_i .

The properties a) and b) permit to extend the whole theory to operators H_m generated by kernels $K(x)$ defined on general local topological abelian groups, instead of R^n , and obtain thus a complete unification of the theory. Hence, the theory of Hilbert transforms and ergodic theorems are special cases of the general theory of operators obtained by convolutions with kernels K_i satisfying a) and b).

G) Finally we still want to mention, in a rather vague and not rigorous form, some considerations which guided us in the solution of the problems A and B mentioned in E). In these problems the limit operator H , whose boundedness we want to establish, commutes with translations and dilatations: $Hg(x) = [Hf](x+t)$ if $g(x) = f(x+t)$, and $Hg(x) = [Hf](\lambda x)$ if $g(x) = f(\lambda x)$. This property, in turn, implies that any linear homogeneous functional (or differential) equation satisfied by f is simultaneously satisfied by Hf . For instance, if $f(x)$ is defined by $f(x) = 1$ if $0 \leq x \leq 1$, $f(x) = 0$ otherwise, then $f(2x) + f(2x-1) = f(x)$, and this equation is clearly satisfied by its Hilbert transform

$$Hf(x) = \log \frac{x}{1-x} \text{ if } 0 < x < 1, \quad Hf(x) = \log \frac{x}{x-1} \text{ otherwise.}$$

Such equations have usually a unique continuous solution. The solution $Hf(x)$ is, however, different from the solution $f(x)$, because

$f(x)$ and $Hf(x)$ have discontinuities at $x = 0$ and $x = 1$, yet they are determined by their values in arbitrary small neighbourhoods of $x = 0$ and $x = 1$, that is, by their «local behaviour».

Thus we may announce the following vague «principle»: If some homogeneous relation between f and Hf is true locally then it is also true everywhere, and the behaviour of $|Hf - f|$ may be detected by its «initial» values. According to this «principle» some relations between f and Hf are more likely to occur, if $f(x)$ is normalized by the condition $\int f(t) dt = 0$. Also, if a certain relation between f and Hf is true locally it may be expected to be true everywhere. This is just the intuitive meaning of the solution of problem A and B given in the precedent papers [7] and [8], where «local» relation between operators, and conditions for normalized function, are considered.

Another consequence of these considerations, is that if Hf and f are related by certain functional relations, we may expect that these relations could give the key to the proof for the properties of Hf .

In the case of the ordinary Hilbert transform there exist simple relations between Hf and f and we will show (§ 6, A) that these relations actually permit a very simple proof for the basic properties of Hf . Since this method of functional relations is related to the operational method of the theory of distributions we think that the above results may be generalized to a certain class of operators of the form $Hf = f \star T$, where T is a distribution.

2. The method of the maximal operator. As we have seen in the Introduction, in the case of Hilbert transform as well as in that of the ergodic theorems, the main object is to establish that a certain sequence of operators $H_i = H_m$, $i = m^{-1}$, possesses the following properties: 1) H_m converges in the p^{th} mean to a limit operator Hf on L^p , if $p > 1$. 2) $H_m f(x)$ is pointwise convergent to $Hf(x)$ for every $f \in L^p$, $p \geq 1$. 3) The limit operator Hf is bounded on L^p , if $p > 1$, and satisfies the inequalities (V) and (VI) of the introduction. 4) The maximal operator $Mf = \sup H_m f$ is bounded on L^p , $p > 1$ and satisfies the inequalities (VIIb) and (VIIc) of the introduction.

On the other hand, in both cases, there exists a certain set $\mathbf{D}$ of «good» functions, dense in L^p , such that the convergence theorems 1) and 2) are immediate for the function $f \in \mathbf{D}$. In the case of the Hilbert transforms, as such good functions, can be taken all the step functions, or the differentiable functions.

First of all we shall show that the problem of proving the four above properties (for general sequences H_m) can be reduced to the considerably simpler problem of proving that the property 4) is true for the functions $f \in \mathbf{D}$. In other words, the truth or the four above properties for all functions depends upon the verification only of the « maximal theorem » 4), and that only for the functions $f \in \mathbf{D}$. This method of using the maximal operator and the « elementary » space $\mathbf{D}$, can be formulated in general terms as follows.

Let $\Omega = \{P\}$ be a measure space, μ a measure on Ω , and $L^p = L^p(\Omega, \mu)$ the set of all real measurable functions $f(P)$, defined on Ω , and such that

$$\|f\|_p = \left\{ \int_{\Omega} |f(P)|^p d\mu \right\}^{1/p} < \infty.$$

For each $p \geq 1$ let $\mathbf{D}_p$ be a fixed set of functions, dense in the space L^p . If $T = Tf = Tf(P) = [Tf](P)$ is some (not necessarily linear) operator defined on a set of functions $\mathbf{D}' = \{f\}$, we shall say (cfr. the precedent paper [7]) that T is of the type p on $\mathbf{D}'$ if

$$\int_{\Omega} |Tf(P)|^p d\mu \leq O_p \int_{\Omega} |f(P)|^p d\mu, \quad (1)$$

for every function $f \in \mathbf{D}'$, O_p being independent of P and f . We shall say that T satisfies condition (p) on $\mathbf{D}'$, if

$$(2) \quad \mu E_{\lambda} = \mu (E[|Tf(P)| \geq \lambda]) \leq \frac{O_p}{\lambda^p} \int_{\Omega} |f(P)|^p d\mu,$$

for every $f \in \mathbf{D}'$ and every $\lambda > 0$.

We have seen in the precedent paper [7] that T satisfies condition (p) if and only if

$$(3) \quad \int_E |Tf|^p d\mu \leq O_p \left\{ \mu E \right\}^{1-\alpha/p} \left\{ \int_{\Omega} |f|^p d\mu \right\}^{\alpha/p}, \quad (0 < \alpha < p)$$

for any set $E \subset \Omega$ of finite measure. We have also seen, that if T satisfies condition (1) and condition (p) , $p > 1$, then

$$\int_E |Tf| d\mu \leq O_E \int_{\Omega} |f| (1 + \log^+ |f|) d\mu + O_E. \quad (4)$$

Thus, an operator T satisfies on $\mathbf{D}'$ the three inequalities (VII a), (VII b) and (VII c) of the introduction, if and only if T satisfies condition (1) and is of the type p , for every $p > 1$.

Let us fix now a set of operators $\{H_\varepsilon\}$, $\varepsilon > 0$, with the following three properties:

A) For each $\varepsilon > 0$, $H_\varepsilon f$ is defined for any $f \in L^p$ and any $p \geq 1$; it assigns to every function $f \in L^p$ another function $H_\varepsilon f(P)$, finite for all $P \in \Omega$, and linear in f :

$$H_\varepsilon(\sum_1^k k_i f_i) = \sum_1^k k_i H_\varepsilon f_i.$$

B) If $f_n \in L^p$ (fixed $p \geq 1$), $n = 1, 2, \dots$, and $\|f_n\|_p \rightarrow 0$ as $n \rightarrow \infty$, then $\lim_{n \rightarrow \infty} H_\varepsilon f_n(P) = 0$, for all $P \in \Omega$, and for each $\varepsilon > 0$.

C) $H_\varepsilon f$ is pointwise convergent on $\mathbf{D}_2$, as $\varepsilon \rightarrow 0$:

$$(5) \quad \lim_{\varepsilon \rightarrow 0} H_\varepsilon f(P) = Hf(P), \text{ for almost all } P \in \Omega, \text{ and all } f \in \mathbf{D}_2.$$

Let Mf be the maximal operator of the sequence $H_\varepsilon f$, defined by

$$Mf(P) = \sup_{\varepsilon > 0} |H_\varepsilon f(P)|.$$

Then by (5), $Mf(P) < \infty$ for every $f \in \mathbf{D}_2$, and almost all $P \in \Omega$, and $|Hf(P)| \leq Mf(P)$.

PROPOSITION 1. Let H be a set of operators satisfying A), B) and C).

a) If the maximal operator $M = Mf$ is of the type p on $\mathbf{D}_p$, $p \geq 1$, then M is of the type p on the whole space L^p .

b) If M satisfies condition (2) on $\mathbf{D}_2$, then it satisfies condition (2) on the whole space $L^2(\Omega)$, and the pointwise convergence (5) holds for all $f \in L^2$.

c) If M satisfies condition (2) on $\mathbf{D}_2$ and condition (p) on $\mathbf{D}_p$, $p \geq 1$, then it satisfies these conditions on L^2 and L^p , and the pointwise convergence (5) holds for every $f \in L^2$ and every $f \in L^p$.

PROOF. a) Assume that M is of the type p on $\mathbf{D}_p$, that is

$$(6) \quad \int_{\Omega} |Mg(P)|^p d\mu \leq O_p \cdot \int_{\Omega} |g(P)|^p d\mu,$$

for every $g \in \mathbf{D}_p$. Let $f \in L^p$. Since $\mathbf{D}_p$ is dense in L^p there is a sequence $g_k \in \mathbf{D}_p$ such that

$$\lim_{k \rightarrow \infty} \|f - g_k\|_p = 0.$$

It is easy to see that M possesses the subadditive property:

$$Mg(P) + Mh(P) \leq M(g+h)(P),$$

for any $g, h \in \mathbf{D}_p$. Hence $\|Mg - Mh\|_p \leq \|M(g-h)\|_p$, and

$$\|g_k - g_{k+i}\|_p = 0$$

implies

$$\|Mg_k - Mg_{k+i}\|_p \leq \|M(g_k - g_{k+i})\|_p \leq O_p \|g_k - g_{k+i}\|_p = 0.$$

Thus,

$$\lim_{k \rightarrow \infty} Mg_{k'}(P) = M_1(P)$$

exists almost everywhere on Ω , where $\{g_{k'}\}$ is a subsequence of $\{g_k\}$. By B), for each $\varepsilon > 0$,

$$|H_\varepsilon f(P)| = \lim_{k \rightarrow \infty} |H_\varepsilon g_{k'}(P)| \leq \lim_{k \rightarrow \infty} Mg_{k'}(P) = M_1(P),$$

therefore $M(P) \leq M_1(P)$, almost everywhere, and by (6)

$$\begin{aligned} \int_{\Omega} |Mf(P)|^p d\mu &\leq \int_{\Omega} |M_1 f(P)|^p d\mu = \lim_{k \rightarrow \infty} \int_{\Omega} |Mg_{k'}(P)|^p d\mu \\ &\leq O_p \cdot \lim_{k \rightarrow \infty} \int_{\Omega} |g_{k'}(P)|^p d\mu = O_p \int_{\Omega} |f|^p d\mu. \end{aligned}$$

Hence M is of the type p on the whole space L^p .

b) Assume now that M satisfies condition (2) on $\mathbf{D}_2$, so that if $E_\lambda(g)$ is the set of points P for which $Mg(P) \geq \lambda$, then

$$\mu \{E_\lambda(g)\} \leq \frac{O_2}{\lambda^2} \int_{\Omega} |g(P)|^2 d\mu.$$

for any $g \in \mathbf{D}_2$. Given $f \in L^2$, there exists a sequence $g_n \in \mathbf{D}_2$, such that $\|f - g_k\|_2 \rightarrow 0$ and $\lim_{k \rightarrow \infty} g_k(P) = f(P)$ almost everywhere, as $k \rightarrow \infty$. If $Mf(P) < \infty$, there is an $\varepsilon > 0$ such that $Mf(P) \leq 2 + H_\varepsilon f(P)$, and by B), $|H_\varepsilon f(P)| \leq 2 + H_\varepsilon g_k(P)$ for all $k > k_0$. Therefore,

$$\begin{aligned} \mu E_\lambda(f) &= \mu \{P : Mf \geq \lambda\} \leq \lim_{k \rightarrow \infty} \mu E_\lambda(g_k) \\ &\leq \lim_{k \rightarrow \infty} \frac{O_2}{\lambda^2} \int_{\Omega} |g_k(P)|^2 d\mu = \frac{O_2}{\lambda^2} \int_{\Omega} |f(P)|^2 d\mu, \end{aligned}$$

Hence M satisfies condition (2) on the whole space L^2 . Now given $f \in L^2$ and $\varepsilon > 0$, there is a $g \in \mathbf{D}_2$ such that $f = g + h$ and $\|h\|_2 \leq \varepsilon^2$. Hence, if E_δ is the set of point P for which $|Mh(P)| > \delta$, then

$$\mu E_\delta \leq \frac{O_2}{\delta^2} \int_{\Omega} |h(P)|^2 d\mu \leq \frac{O_2}{\delta^2} \cdot \delta^4 = O_2 \cdot \varepsilon^2,$$

and $\mu E_\delta \rightarrow 0$, as $\varepsilon \rightarrow 0$. For P not belonging to E_δ , we have

$$|H_\varepsilon h(P)| \leq |Mh(P)| < \delta,$$

for all $\varepsilon > 0$. Therefore, since $g \in \mathbf{D}_2$, and by (5), given P not belonging to E_δ there is an ε_0 such that

$$|H_{\varepsilon_1} g(P) - H_{\varepsilon_2} g(P)| < \delta, \quad \text{and} \quad |H_{\varepsilon_1} h(P) - H_{\varepsilon_2} h(P)| < 2\delta,$$

for any $\varepsilon_1, \varepsilon_2 > \varepsilon_0$. Hence $|H_{\varepsilon_1} f(P) - H_{\varepsilon_2} f(P)| < 3\delta$ for P not belonging to E_δ and $\varepsilon_1, \varepsilon_2 > \varepsilon_0$. Since the measure of E_δ may be taken arbitrarily small, it follows that $H_\varepsilon f(P)$ converges for almost all $P \in \Omega$, as $\varepsilon \rightarrow 0$.

The proof of c) is quite similar.

COROLARY. If the maximal operator $M = Mf$ satisfies condition (1) on $\mathbf{D}_1$ and is of the type p on $\mathbf{D}_p$, for every $p < j$, then

- 1) M satisfies condition 1 on L^1 and is of the type p on L^p , for $p \geq 1$.
- 2) $H_\varepsilon f(P)$ converges almost everywhere to $Hf(P)$, for every $f \in L^p$, $p \geq 1$.
- 3) $H_\varepsilon f$ converges in the p^{th} mean to Hf , for every $f \in L^p$ and every $p > 1$.
- 4) For every set E of finite measure,

$$\lim_{\varepsilon \rightarrow 0} \int_E |Hf(P) - H_\varepsilon f(P)|^q d\mu = 0, \quad \text{if } f \in L^1 \text{ and } 0 < q < 1,$$

$$\lim_{\varepsilon \rightarrow 0} \int_E |Hf(P) - H_\varepsilon f(P)| d\mu = 0 \quad \text{if } |f| \log^+ |f| \in L^1.$$

The last two inequalities are consequences of Proposition I and of theorem 2 of the precedent paper [7].

REMARK 1. Proposition I asserts that if M satisfies condition (p) then the pointwise and the mean convergence of the H_ε hold. This can be generalized as follows.

Let $\tilde{H}f$ be the operator

$$\tilde{H}f(P) = \limsup_{\varepsilon \rightarrow 0} H_\varepsilon f(P) - \liminf_{\varepsilon \rightarrow 0} H_\varepsilon f(P),$$

so that $|\tilde{H}f(P)| \leq 2Mf(P)$.

By a theorem of S. Saks (the idea of the proof is given in Proposition 1a, below), if $\mu(\Omega) < \infty$, and if $Mf(P) < \infty$ almost everywhere, for all $f \in L^p$, then $\|f - g_k\|_p \rightarrow 0$ implies

$$|\tilde{H}f(P) - \tilde{H}g_{k'}(P)| \rightarrow 0,$$

for almost all P , where $\{g_{k'}\}$ is a subsequence of $\{g_k\}$.

From this theorem of Saks, and from condition B), it follows that in order to establish the pointwise and the mean convergence of $H_\varepsilon f$, it is enough to prove that $Mf(P) < \infty$ almost everywhere, for all $f \in L^p$.

This argument is due to K. Yosida [15], who used it in the proof of the ergodic theorem.

The argument of Saks-Yosida can be used to prove also the following proposition.

PROPOSITION 1a. *Let $\mu(\Omega) < \infty$, and let $L_s^* \subset L^1$, ($s \geq 1$), be the set of all functions $f(P)$ such that*

$$\|f\|_s = \int_{\Omega} |f(P)| (1 + \log^+ |f(P)|)^s d\mu < \infty.$$

If $Mf(P) < \infty$ almost everywhere, for every $f \in L_s^$, and if the pointwise convergence (5) holds for every $f \in L^2$, then (5) holds for every $f \in L_s^*$.*

PROOF. Let (S) be the space of all the measurable functions $f(P)$ such that $|f(P)| < \infty$ almost everywhere, and let

$$\|f\|_s = \int_{\Omega} \frac{|f(P)|}{1 + |f(P)|} d\mu.$$

It is well known that (S) with the metric $\|f\|_s$ is a linear space of type (F) , and the convergence in the metric $\|f\|_s$ is equivalent to

the convergence in measure. If $\widetilde{H}f$ is defined as above, then by assumption $\widetilde{H}f(P) \leq 2Mf(P) < \infty$ almost everywhere for every $f \in L_s^*$. Therefore $\widetilde{H}f$ assigns to each $f \in L_s^*$ a function $\widetilde{H}f \in (S)$. Let Σ be the set of all the functions $f \in L_s^*$ such that $\|f\|_* \leq 1$. Each $f \in \Sigma$ belongs to L^1 , and by Fatou's lemma it is easy to see that Σ , as a subset of L^1 , is closed in L^1 . Therefore Σ , with the metric of L^1 , is a complete metric space. From B) it follows that for each $\varepsilon > 0$, $H_\varepsilon f$ is a continuous transformation of Σ into (S) (Σ with the metric of L^1). By the category theorem of Baire (cfr. Yosida [15]), there exists a constant C and an open set U of L^1 , such that $U \cap \Sigma$ is a non-void set and $\|\widetilde{H}f\|_s \leq C$ for every $f \in U \cap \Sigma$.

Let $f \in U \cap \Sigma$ and take $\delta > 0$ so small that $f_0(1-\delta)f \in U \cap \Sigma$.

For $\gamma > 0$ sufficiently small, $\|g\|_* < \gamma^2$ implies $f_0 + g \in U \cap \Sigma$. In fact, if E is the set where $|g(P)| \leq \gamma |f_0(P)|$ then

$$\begin{aligned} |f_0| + |g| + (|f_0| + |g|) \log^+ (|f_0| + |g|) \\ \leq (1+\gamma)(|f_0| + |f_0| \log^+ |f_0|) + 2\gamma(1+\gamma)|f_0| \end{aligned}$$

on E , and

$$\leq 2|\gamma^{-1}(|g| + |g| \log^+ |g|) + \gamma^{-1} \log(\gamma^{-1})|g||,$$

on $\Omega \setminus E$. Hence,

$$\begin{aligned} \|f_0 + g\|_* &\leq (1+\gamma)\|f_0\|_* + 2|\gamma^{-1}\|g\|_* + \gamma\|f_0\|_* + \gamma^{-1} \log(\gamma^{-1})\|g\|_* \\ &\leq (1+\gamma)\|f_0\|_* + 6\gamma + 2\gamma \log(\gamma^{-1}) \leq 1, \end{aligned}$$

if γ is sufficiently small.

Therefore $\|\widetilde{H}g\|_s \leq \|\widetilde{H}(f_0 + g)\|_s + \|\widetilde{H}f_0\|_s \leq 2C$, for every g such that $\|g\|_* \leq \gamma^2$. From this and from $|\widetilde{H}(\lambda f)| \leq 2|\lambda| |\widetilde{H}f|$, it is easy to deduce that $\|g_i\|_* \rightarrow 0$ implies $\|\widetilde{H}g_i\|_s \rightarrow 0$.

Now, given $f \in L_s^*$, there exists a sequence $h_i \in L_s^* \cap L^2$ such that $|h_i(P)| \leq |f(P)|$ and $\|h_i - f\|_1 \rightarrow 0$. From $|h_i| \leq |f|$ it follows that

$$|h_i - f| + |h_i - f| \log^+ |h_i - f| \leq Cst \cdot |f| (1 + \log^+ |f|),$$

and applying to the Lebesgue theorem on term by term integration, we obtain that $\|f - h_i\|_* \rightarrow 0$. By assumption, $h_i \in L^2$ implies $\widetilde{H}h_i = 0$

almost everywhere, hence $\|\tilde{H}f\|_s \leq \|\tilde{H}h_i\|_s + \|\tilde{H}(f - h_i)\|_s = \|\tilde{H}(f - h_i)\|_s \rightarrow 0$, as $i \rightarrow \infty$.

Thus, $\|\tilde{H}f\|_s = 0$ for every $f \in L_s^*$, and this proves the proposition.

3. Auxiliary lemmas. Let $K(x)$ be a function, defined on the n -dimensional euclidean space $R^n = \{x\}$ such that:

$\alpha)$ $K(x) = 0$ if $|x| > 2$, where $|x|$ denotes the distance of x to the origin.

$$\beta) \int_{R^n} K(x) dx = 0.$$

$$\gamma) \int_{R^n} |K(x-t) - K(x)| dx \leq O_1 \cdot |t|, \quad (O_1 = \text{constant}).$$

$$\delta) |K(x)| \leq O_1, \text{ for all } x \in R^n.$$

The condition $\delta)$ may be replaced by more general ones, but in order to avoid complications we shall stick to condition $\delta)$.

For each $i = \pm 1, \pm 2, \pm 3, \dots$ we define

$$(7) \quad K_i(x) = \frac{1}{2^{ni}} K\left(\frac{x}{2^i}\right),$$

where n is the dimension of the space $R^n = \{x\}$. From the conditions $\alpha) - \delta)$ we obtain

$$\alpha_i) K_i(x) = 0 \text{ if } |x| > 2^i,$$

$$\beta_i) \int_{R^n} K_i(x) dx = 0,$$

$$\gamma_i) \int_{R^n} |K_i(x) - K_i(x-t)| dx \leq O_1 \cdot \frac{|t|}{2^i},$$

$$\delta_i) |K_i(x)| \leq \frac{O_1}{2^{ni}}, \quad \|K_i\|_1 = \int_{R^n} |K_i(x)| dx \leq O_1'.$$

Let $\Omega = \{P\}$ be, as before, a measure space with a measure μ , and let $\sigma_x, x \in R^n$, be a n -parametric group of measure preserving transformations on Ω : For each $x \in R^n$, $\sigma_x P$ is an one to one transformation, assigning to each point $P \in \Omega$ another point $\sigma_x P \in \Omega$, in such a way that $(x+y)$ denotes the vector sum of x and y

$$\sigma_{x+y}P = \sigma_x(\sigma_yP) = \sigma_y(\sigma_xP), \quad \sigma_0P = P,$$

and such that for each set $E \subset \Omega$,

$$(8) \quad \mu(\sigma_x E) = \mu(E),$$

where $\sigma_x E$ is that of all the points of the form $\sigma_x P$, $P \in E$. To any function $f(P)$, defined and measurable on Ω , we assign the function

$$(9) \quad f(P, t) = f(\sigma_t P)$$

defined on $\Omega \times R^n$. We assume that $\{\sigma_x\}$ satisfies the usual measurability conditions so that if $f(P)$ is a measurable function on Ω then $f(P, x)$ is a measurable function on the product measure space $\Omega \times R^n$. From (8) it follows then, that

$$(8a) \quad \int_{\Omega} f(P, x) d\mu = \int_{\Omega} f(P, x) d\mu(P) = \int_{\Omega} f(P) d\mu, \text{ for every } x \in R^n$$

If f is μ -integrable on Ω (or if $f \in L^p(\Omega, \mu)$, $p \geq 1$), then $f(P, x)$, as a function of x , is integrable on every finite n -dimensional sphere $B = B(r) = \{ |x| \leq r \}$, for almost all P . In fact, by (8a) and by Fubini's theorem,

$$\begin{aligned} \int_{\Omega} \left\{ \int_B |f(P, x)| dx \right\} d\mu(P) &= \int_B \left\{ \int_{\Omega} |f(P, x)| d\mu(P) \right\} dx \\ &= \int_B \left\{ \int_{\Omega} |f(P)| d\mu \right\} dx = V(r) \cdot \int_{\Omega} |f(P)| d\mu < \infty, \end{aligned}$$

where $V(r)$ is the volume of the sphere $B(r)$. Therefore,

$$\int_B |f(P, x)| dx < \infty$$

for almost all $P \in \Omega$.

Since $K_i(x) = 0$ for $|x| > 2^i$, it follows that the integral

$$(10) \quad T_i f(P) = \int_{R^n} f(P, t) K_i(t) dt = \int_{B(2^i)} f(P, t) K_i(t) dt$$

exists for almost all P , and defines an operator T_i on Ω , for every $i = \pm 1, \pm 2, \dots$

For each $m = 1, 2, \dots$ we define the operators

$$(10 a) \quad H_m = \sum_{i=-m}^m T_i,$$

$$(10 b) \quad H_m^+ = \sum_{i=0}^m T_i, \quad H_m^- = \sum_{i=-m}^{-1} T_i = H_m - H_m^+.$$

Similarly we define on the space $R^n = \{x\}$ the operators :

$$(11) \quad T_i g(x) = (g \star K_i)(x) = \int_{R^n} g(x+t) K_i(t) dt,$$

$$(11 a) \quad \overline{H}_m = \sum_{i=-m}^m \overline{T}_i,$$

and similarly are defined $\overline{H}_m^+$, $\overline{H}_m^-$.

Since

$$(T_i f)(\sigma_x P) = T_i f(P, x) = \int_{R^n} f(P, x+t) K_i(t) dt = [f(P, \cdot) \star K_i](x),$$

each operator T_i , or H_m , assigns to each function $f(P)$, defined on Ω , a function $(T_i f)(P, x)$ defined on $\Omega \times R^n$. For fixed P , $T_i f(P, x) = \overline{T}_i g(x) = [\overline{T}_i f(P, \cdot)](x)$, where $g(x) = f(P, x)$.

And for

$$x=0, \quad T_i f(P, 0) = T_i f(P).$$

We shall first deduce some simple lemmas concerning the kernels $K(x)$.

For each function $g(x)$ defined on R^n we will denote by $g(x)_{(N)}$ $= g_{(N)}(x)$ the function equal to $g(x)$ if $|x| \leq N$, and zero otherwise. In particular $f(P, x)_{(N)} = f(P, \cdot)_{(N)}(x)$ is equal to $f(P, x)$ if $|x| \leq N$, and zero otherwise. We will use the notation $\varphi(x) = \varphi(x, N)$ for a function $\varphi(x)$ such that $\varphi(x) = \varphi(x)_{(N)}$. We will also denote the sphere $|t-x| \leq r$ by $B(r) = B(x, r)$, and its volume by $V(r)$. Finally $\varphi_a(x)$, $\varphi_{a,u}(x)$, $x, u \in R^n$, will denote the functions defined on $R^n = \{x\}$ by

$$(12) \quad \varphi_{a,u}(x) = \begin{cases} (V(a))^{-1} & \text{if } |u-x| \leq a, \\ 0 & \text{otherwise} \end{cases}, \quad \varphi_a(x) = \varphi_{a,0}(x),$$

so that

$$(12 a) \quad \int_{R^n} \varphi_{a,u}(x) dx = \|\varphi_{a,u}\|_1 = \|\varphi_a\|_1 = 1.$$

We will use frequently the following lemma, whose proof is obvious.

LEMMA 1. If $\varphi(x) = \varphi(x, N)$ vanishes for $|x| > N$, then

$$(13) \quad \int_{R^n} f(P, x+t) \varphi(t) dt = \int_{R^n} f(P, x+t)_{(2N)} \varphi(t) dt \\ = [f(P, \cdot)_{(2N)} \star \varphi](x),$$

for each x of the sphere $|x| < N$. In particular

$$(14) \quad T_i f(P, x) = \overline{T}_i [f(P, \cdot)_{(2^i)}](x) = \int_{R^n} f(P, x+t)_{(2^i)} K_i(t) dt,$$

for all x of the sphere $|x| \leq 2^i$.

LEMMA 2. Let $\varphi(x) = \varphi(x, N)$ be a function vanishing for $|x| > N$ and such that

$$\int_{R^n} \varphi(x) dx = 0.$$

Then

$$(15) \quad \|\varphi \star K_i\|_1 \leq \frac{O_1 \cdot N}{2^i} \|\varphi\|_1,$$

where

$$\varphi \star K_i(x) = \int_{R^n} K_i(x-t) \varphi(t) dt.$$

In particular

$$(15a) \quad \|K_i \star K_j\|_1 \leq O_1 \cdot 2^{-|i-j|}.$$

PROOF. In fact, by assumption and by γ_1 ,

$$\|\varphi \star K_i\|_1 = \int_{R^n} \left| \int_{R^n} \varphi(t) K_i(x-t) dt \right| dx = \int_{R^n} \left| \int_{R^n} \varphi(t) \{ K_i(x-t) - \right. \\ \left. - K_i(x) \} dt \right| dx \\ \leq \int_{R^n} |\varphi(t)| \left\{ \int_{R^n} |K_i(x-t) - K_i(x)| dx \right\} dt \\ \leq \sup_{|t| \leq N} \left\{ \int_{R^n} |K_i(x-t) - K_i(x)| dx \right\} \cdot \int_{R^n} |\varphi(t)| dt \\ \leq \frac{O_1 \cdot N}{2^i} \|\varphi\|_1.$$

In particular, since $K_j(x)$ vanishes for $|x| > 2^j$, and by β_1) its integral is zero, we obtain (15 a).

LEMMA 3. Let $\varphi(x)$, $x \in \mathbb{R}^n$, be a function of the form $\varphi(x) = \varphi_{a,u}(x) - \varphi_{b,v}(x)$, where $\varphi_{a,u}(x)$ is defined by (12). Then :

$$(16) \quad \|\varphi * K_i\|_1 \leq \frac{O_1(a+b+|u|+|v|)}{2^i}, \text{ if } i > 0,$$

$$(17) \quad \|\varphi * K_i\|_1 \leq O_1 2^i ([V(a)]^{-1} + [V(b)]^{-1}), \text{ if } i < 0.$$

PROOF. By (12 a), $\|\varphi\|_1 \leq 2$, and φ vanishes for $|x| > a+b$, and its integral is zero. Hence (16) is a particular case of (15). In order to prove (17) we may assume $u=v=0$. Since $\varphi_a(x)$ is constant on the sphere $|x| < a$ and on the sphere $|x| > a$, and since the integral of K_i is zero, it is easy to see that the function $\varphi_a * K_i(x)$ vanishes for any point x such that $||x| - a| > 2^i$. Since

$$\varphi_a \leq (V(a))^{-1} [\varphi_a * K_i(x)] \leq (V(a))^{-1} \cdot \|K_i\|_1 = (V(a))^{-1},$$

we obtain

$$\|\varphi_a * K_i\|_1 \leq 2^i (V(a))^{-1},$$

and similarly $\|\varphi_b * K_i\|_1 \leq 2^i (V(b))^{-1}$, and this proves (17)

LEMMA 4. Let $\varphi(x)$, $x \in \mathbb{R}^n$, be a function vanishing for $|x| > N$ and let

$$\Phi f(P) = \int_{\mathbb{R}^n} f(P, t) \varphi(t) dt.$$

Then

$$(18) \quad \|\Phi f\|_{p^p} = \left\| \left| \Phi f(P) \right|^p d\mu(P) \right\|_1 \leq O_p \|\varphi\|_1^p \cdot \|f\|_{p^p},$$

for $p \geq 1$.

PROOF. Taking in account the measure perserving property of σ_x and lemma 1, we shall have

$$\begin{aligned} \int_{\mathbb{R}^n} |\Phi f(P)|^p d\mu(P) &= \frac{1}{V(N)} \int_{B(N)} \left\{ \int_{\mathbb{R}^n} |\Phi f(P, x)|^p d\mu(P) \right\} dx \\ &= \frac{1}{V(N)} \int_{B(N)} \left\{ \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(P, x+t) \varphi(t) dt \right|^p d\mu(P) \right\} dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{V(N)} \int_{\Omega} \left\{ \int_{R(N)} \left| \int_{R^n} f(P, x+t) \varphi(t) dt \right|^p dx \right\} d\mu(P) \\
 &= \frac{1}{V(N)} \int_{\Omega} \left\{ \int_{R(N)} \left| \int_{R^n} f(P, x+t)_{(2N)} \varphi(t) dt \right|^p dx \right\} d\mu(P) \\
 &\leq \frac{1}{V(N)} \int_{\Omega} \left\{ \int_{R^n} \left| \int_{R^n} f(P, x+t)_{(2N)} \varphi(t) dt \right|^p dx \right\} d\mu(P) \\
 &= \frac{1}{V(N)} \int_{\Omega} \left\{ \int_{R^n} |h(P, x)|^p dx \right\} d\mu(P),
 \end{aligned}$$

where $h(P, x)$ is defined by

$$h(P, x) = \int_{R^n} f(P, x+t)_{(2N)} \varphi(t) dt = [f(P, \cdot)_{(2N)} * \varphi](x).$$

By a known inequality of Young ([1], p. 76),

$$\begin{aligned}
 \int_{R^n} |h(P, x)|^p dx &\leq \|\varphi\|_1^p \cdot \int_{R^n} |f(P, t)_{(2N)}|^p dt \\
 &= \|\varphi\|_1^p \int_{R(2N)} |f(P, t)|^p dt.
 \end{aligned}$$

Hence we obtain,

$$\begin{aligned}
 \int_{\Omega} |\Phi f(P)|^p d\mu &\leq \frac{1}{V(N)} \int_{\Omega} \left\{ \|\varphi\|_1^p \cdot \int_{R(2N)} |f(P, t)|^p dt \right\} d\mu(P) \\
 &= \frac{\|\varphi\|_1^p}{V(N)} \int_{R(2N)} \left\{ \int_{\Omega} |f(P, t)|^p d\mu(P) \right\} dt \\
 &= \frac{V(2N)}{V(N)} \|\varphi\|_1^p \int_{\Omega} |f(P)|^p d\mu = O_p \|\varphi\|_1^p \cdot \|f\|_p^p.
 \end{aligned}$$

LEMMA 5. If $h(x)$ is defined on $R^n = \{x\}$ and $h \in L^p(R^n)$, $p > 1$, then

$$\lim_{\varepsilon \rightarrow 0} \|h - (h * \varphi_\varepsilon)\|^p = 0,$$

where

$$(h \star \varphi_\varepsilon)(u) = \int_{R^n} h(t) \varphi_{\varepsilon, u}(t) dt = \frac{1}{V(\varepsilon)} \int_{|t-u| < \varepsilon} h(t) dt.$$

PROOF. This is a well known theorem and is an immediate consequence of the Hardy-Littlewood maximal theorem (see the precedent paper [8]). In fact, by the Lebesgue theorem

$$\lim_{\varepsilon \rightarrow 0} (h \star \varphi_\varepsilon)(u) = \lim_{\varepsilon \rightarrow 0} \frac{1}{V(\varepsilon)} \int_{|t-u| < \varepsilon} h(t) dt = h(u).$$

for almost all $u \in R^n$, so that

$$\lim_{\varepsilon \rightarrow 0} \|h(x) - (h \star \varphi_\varepsilon)(x)\|^p = 0$$

for almost all x . By Hardy-Littlewood's theorem $|h \star \varphi_\varepsilon(x)| \leq \Lambda(x)$, where $\Lambda \in L^p$, if $p < 1$. Hence we may integrate term by term and

$$\lim_{\varepsilon \rightarrow 0} \int_{R^n} |h(x) - (h \star \varphi_\varepsilon)(x)|^p dx = 0,$$

and this proves the lemma.

LEMMA 6. If $f \in L^p(\Omega, \mu)$, and $p < 1$, then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|f(\sigma_\varepsilon P) - (f(P, \cdot) \star \varphi_\varepsilon)(u)\|_p = \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| f(P, u) - \int_{R^n} f(P, t) \varphi_{\varepsilon, u}(t) dt \right|^p d\mu = 0, \end{aligned}$$

for every $u \in R^n$.

PROOF. Let $\varepsilon < 1$, $\varepsilon \rightarrow 0$. We may assume $u = 0$. Taking in account the measure perserving property of σ_ε , lemma 1, and that $\varphi_\varepsilon(t) = 0$ for $|x| > \varepsilon$, we shall have:

$$\begin{aligned} \int_{\Omega} \left| f(P) - \int_{R^n} f(P, t) \varphi_\varepsilon(t) dt \right|^p d\mu(P) = \\ \frac{1}{V(1)} \int_{|x| \leq 1} \int_{\Omega} \left| f(P, x) - \int_{R^n} f(P, t+x) \varphi_\varepsilon(t) dt \right|^p d\mu(P) dx \end{aligned}$$

$$= \frac{1}{V(1)} \int_{\Omega} \left\{ \int_{|x| \leq 1} \left| \int_{R^n} f(P, x)_{(2)} - \int_{R^n} f(P, t+x)_{(2)} \varphi_\varepsilon(t) dt \right|^p dx \right\} d\mu(P) \\ \leq \frac{1}{V(1)} \int_{\Omega} \|f(P, \cdot)_{(2)} - [f(P, \cdot)_{(2)} * \varphi_\varepsilon]\|_p^p d\mu(P).$$

By the preceding lemma the last integral tends to zero for $\varepsilon = 0$, and by Hardy-Littlewood's maximal theorem this integral is dominated by a function $F(P)$ of the form

$$F(P) \leq O_p \int_{R^n} |f(P, t)_{(2)}|^p dt = O_p \int_{|t| \leq 2} |f(P, t)|^p dt.$$

Since $F(P)$ is an integrable function [because

$$\int_{\Omega} F(P) dP \leq O_p \int_{|t| \leq 2} \int_{\Omega} |f(P, t)|^p d\mu \int dt = O_p \cdot V(2) \int_{\Omega} |f|^p d\mu,$$

we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| f(P) - \int_{R^n} f(P, t) \varphi_\varepsilon(t) dt \right|^p d\mu(P) = 0,$$

and this proves the lemma.

DEFINITION 1. We will denote by $\mathbf{D}_2' = \mathbf{D}_2'(\Omega)$ the set of all functions $f(P) \in L^2(\Omega, \mu)$ of the form

$$f(P) = \int_{R^n} g(P, t) (\varphi_{\varepsilon, u}(t) - \varphi_{\varepsilon, u'}(t)) dt \\ = g(P, \cdot) * \varphi_{\varepsilon, u} - g(P, \cdot) * \varphi_{\varepsilon, u'},$$

where $g \in L^2 \cap L^1$.

We will denote by $\mathbf{D}_2''$ the set of all functions $f(P) \in L_2(\Omega, \mu)$ which are σ_x -invariant, that is, $f(\sigma_x P) = f(P)$ for all $x \in R^n$ and almost all $P \in \Omega$. Finally we denote by $\mathbf{D}_2 = \mathbf{D}_2' + \mathbf{D}_2''$ the set of all functions of the form $f = f' + f''$, where $f' \in \mathbf{D}_2'$, $f'' \in \mathbf{D}_2''$.

LEMMA 7. The set $\mathbf{D}_2$ is dense in $L^2(\Omega, \mu)$.

PROOF. It is enough to prove that if $g \in L^2(\Omega, \mu)$ is orthogonal to $\mathbf{D}_2'$, then g belongs to $\mathbf{D}_2''$. If g is orthogonal to $\mathbf{D}_2'$, it is orthogonal to any function of the form

$$f(P, \cdot) * \varphi_{\varepsilon, u} - f(P, \cdot) * \varphi_{\varepsilon, u'}, \quad f \in L^2 \cap L^1.$$

Allowing ε tend to zero, we deduce from lemma 6, that g is ortogonal to any function of the form $f(P) - f(\sigma_u P)$, $f \in L^2(\Omega, \mu)$, $u \in K^n$. In particular $(g, g - g_u) = 0$, where $g_u(P) = g(\sigma_u P)$. Besides, by the measure perveing property of σ_x , $(g_u, g_u) = (g, g)$. Therefore

$$(g - g_u, g - g_u) = (g_u, g_u) - (g, g) + 2(g, g - g_u) = 2(g, g - g_u) = 0,$$

and $g = g_u$, for every $u \in K^n$. Hence g is invariant, that is $g \in \mathbf{D}_2''$ and this proves the lemma.

3. Proof of the mean and pointwise convergence in L^2 . In this section we shall prove the convergence of the operators $H_m f$ for $f \in L^2$.

Though we will consider the real space $L^2(\Omega, \mu)$ composed by real functions, the results obviously extend to complex functions.

PROPOSITION II. For each i , the operator T_i , defined by (10) assigns to each function $f \in L^2(\Omega, \mu)$ another function $T_i f \in L^2(\Omega, \mu)$ of the same space with the following properties :

1) T_i is a linear operator on the real Hilbert space $L^2(\Omega, \mu)$, and the adjoint operator T_i^* is given by the formula

$$(19) \quad T_i^* f(P) = \int_{K^n} f(P, t) K_i(-t) dt.$$

2) The operators T_i are uniformly bounded on L^2 :

$$(20) \quad \|T_i f\|_2 \leq O_2 \cdot \|f\|_2 \quad (O_2 \text{ independent of } i \text{ and of } i).$$

3) The operators T_i and T_j^* commute :

$$T_i T_j = T_j T_i, \quad T_i T_j^* = T_j^* T_i.$$

4) For any i, j , and any $f \in L^2(\Omega, \mu)$, it is true that

$$(21) \quad \|T_i T_j f\|_2 \leq O_2 \cdot 2^{-|i-j|} \cdot \|f\|_2$$

$$(21^*) \quad \|T_i T_j^* f\|_2 \leq O_2 \cdot 2^{-|i-j|} \cdot \|f\|_2.$$

PROOF. 1) By the measure perserving property of σ_x , we have for every $f, g \in L^2(\Omega, \mu)$,

$$(T_i f, g) = \int_{\Omega} g(P) \left\{ \int_{K^n} f(P, t) K_i(t) dt \right\} d\mu(P) =$$

$$\begin{aligned} \int_{R^n} K_i(t) \left\{ \int_{\Omega} g(P) f(P, t) d\mu \right\} dt &= \int_{R^n} K_i(t) \left\{ \int_{\Omega} g(P-t) f(P) d\mu(P) \right\} dt \\ &= \int_{\Omega} f(P) \left\{ \int_{R^n} g(P, t) K_i(-t) dt \right\} d\mu = (f, T_i^* g). \end{aligned}$$

2) In view of the properties β_1) and γ_1), (20) is an immediate consequence of lemma 4.

3) We have

$$\begin{aligned} T_i(T_j f)(P) &= T_i \left(\int_{R^n} f(P, t) K_j(t) dt \right) = \int_{R^n} \left\{ \int_{R^n} f(\sigma_x P, t) K_j(t) K_i(x) dx \right\} \\ &= \int_{R^n} \left\{ \int_{R^n} f(P, x+t) K_j(t) K_i(x) dt dx \right\} = T_j(T_i f)(P), \end{aligned}$$

and similarly $T_i T_j^* = T_j^* T_i$.

4) By the precedent equality

$$\begin{aligned} T_i T_j f(P) &= \int_{R^n} \left\{ \int_{R^n} f(P, x+t) K_j(t) K_i(x) dt dx \right\} \\ &= \int_{R^n} \left\{ \int_{R^n} f(P, s) K_j(s-x) dt \right\} K_i(x) dx = \int_{R^n} f(P, s) \varphi(s) ds, \end{aligned}$$

where $\varphi(s) = K_j * K_i(s) = \int_{R^n} K_j(s-x) K_i(x) dx$.

By α_1), $\varphi(s)$ vanishes for $|s| > 2^{i+j}$ and by lemma 2 (15a), $\|\varphi\|_1 = \|K_j * K_i\|_1 \leq O_1 2^{-|i-j|}$. Hence, by lemma 4

$$\begin{aligned} \int_{\Omega} |T_i T_j f(P)|^2 d\mu(P) &\leq O_2 \cdot \|\varphi\|_1^2 \cdot \int_{\Omega} |f(P)|^2 d\mu(P) \\ &\leq (O_1 O_2 2^{-|i-j|})^2 \|f\|_2^2 = (O_2' 2^{-|i-j|} \|f\|_2)^2. \end{aligned}$$

This proves (21), and the inequality (21*) is proved in quite a similar way.

PROPOSITION III. For each function $f \in \mathbf{D}_2(\Omega)$ (see Definition 1), the sequences $H_m^+ f(P)$ and $H_m f(P)$ converge in the mean and pointwise that is

$$(22) \quad \lim_{m \rightarrow \infty} H_m f(P) = Hf(P), \quad \lim_{m \rightarrow \infty} H_m^+ f(P) = H^+ f(P)$$

exists for almost all P , and

$$(23) \quad \lim_{m \rightarrow \infty} \int_{\Omega} |Hf(P) - H_m f(P)|^2 d\mu = 0.$$

PROOF. If $f \in \mathbf{D}_2''$, then f is σ_x -invariant, $f(\sigma_x P) = f(P, x) = f(P)$, and by β_1 ,

$$T_i f(P) = \int_{\mathbb{R}^n} f(P, x) K_i(x) dx = f(P) \int_{\mathbb{R}^n} K_i(x) dx = 0,$$

$$H_m f(P) = 0,$$

so that (22) and (23) are true for $f \in \mathbf{D}_2''$.

Let now $g \in \mathbf{D}_2'$. Then, there is a function $f \in L^2(\Omega, \mu) \cap L^1(\Omega, \mu)$ such that

$$\begin{aligned} g(P) &= f(P, \cdot) * \rho_{\varepsilon, u} - f(P, \cdot) * \rho_{\varepsilon, w} \\ &= \int_{\mathbb{R}^n} f(P, t) (\rho_{\varepsilon, u}(t) - \rho_{\varepsilon, w}(t)) dt. \end{aligned}$$

If the function $\rho_{\varepsilon, u} - \rho_{\varepsilon, w}$ is denoted by φ , then

$$\begin{aligned} (24) \quad T_i g(P) &= \int_{\mathbb{R}^n} g(P, t) K_i(t) dt = \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(P, x+t) \varphi(t) dt \right\} K_i(x) dx \\ &= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(P, s) \varphi(s-x) ds \right\} K_i(x) dx \\ &= \int_{\mathbb{R}^n} f(P, x) [\varphi * K_i](x) dx. \end{aligned}$$

Let $|u| + |u'| + \varepsilon + \varepsilon' + (V(\varepsilon))^{-1} + (V(\varepsilon'))^{-1} = a$. By lemma 3, $\|\varphi * K_i\|_1 \leq O_i 2^{-1} a$, if $i > 0$. Therefore from (24) and lemma 4, we obtain

$$\begin{aligned} (25) \quad \int_{\Omega} |T_i g(P)| d\mu &\leq \frac{O_i a}{2^i} \int_{\Omega} |f(P)| d\mu, \quad (i > 0), \\ \int_{\Omega} |T_i g(P)|^2 d\mu &\leq \left(\frac{O_i a}{2^i} \right)^2 \int_{\Omega} |f(P)|^2 d\mu, \quad (i > 0). \end{aligned}$$

By the same lemmas we have, for $i < 0$,

$$(25') \quad \int_{\Omega} |T_i g(P)| d\mu \leq O_i \cdot a \cdot 2^i \int_{\Omega} |f| d\mu, \quad (i < 0),$$

$$(26') \quad \int_{\Omega} |T_i g(P)| d\mu \leq (O_1 a \cdot 2^i)^2 \int_{\Omega} |f|^2 d\mu, \quad (i < 0).$$

Hence

$$(27) \quad \sum_{i=-m}^m \int_{\Omega} |T_i g(P)| d\mu \leq \left(\sum_{i=-m}^0 2^i + \sum_0^m 2^{-i} \right) O_1 a \int_{\Omega} |f(P)| d\mu \\ \leq O_1' a \int_{\Omega} |f| d\mu.$$

$$(27a) \quad \left\{ \int_{\Omega} |H_{m+k} g(P) - H_m g(P)|^2 d\mu \right\}^{1/2} \\ \leq \left(\sum_{i=m+k}^{m+k} 2^{-i} + \sum_{i=-(m+k)}^{-m} 2^i \right) \cdot O_1 \cdot a \left\{ \int_{\Omega} |f(P)|^2 d\mu \right\}^{1/2} \\ \leq \frac{O_2 a}{2^m} \left\{ \int_{\Omega} |f|^2 d\mu \right\}^{1/2}.$$

By (27) the series of non-negative elements $\sum |T_i g(P)|$ converges to a limite function which is integrable, and hence finite almost everywhere. Therefore $\sum T_i g(P)$ is absolutely convergent for almost all P , and $\lim H_m g(P) = \lim \sum_{i=-m}^m T_i g(P)$ exists for almost all P . This proves (22) for $g \in \mathbf{D}'_2$. By (27a), also (23) is true for $g \in \mathbf{D}'_2$. Thus (22) and (23) hold for any $f \in \mathbf{D}'_2$, and any $f \in \mathbf{D}''$, hence for any $f \in \mathbf{D}'_2 + \mathbf{D}''$. This proves the Proposition III.

DEFINITION 2. We define the maximal operators M_N and M by

$$(28) \quad M_N f(P) = \sup_{m \leq N} |H_m f(P)|$$

$$(28a) \quad M f(P) = \sup_{m < \infty} |H_m f(P)| = \lim_{N \rightarrow \infty} M_N f(P).$$

Similarly, if $\bar{H}_m$ are the operators defined by (11a), $x \in \mathbb{R}^n$,

$$(29) \quad \bar{M}_N h(x) = \sup_{m \leq N} |\bar{H}_m h(x)|,$$

$$(29a) \quad \bar{M} h(x) = \lim_{N \rightarrow \infty} \bar{M}_N h(x).$$

$E[|M(P)| \geq \lambda]$ will denote the set of points P for which $|M(P)| \geq \lambda$, and $|E|$ will denote the measure of the set E .

$\bar{M}^+$ and M^+ are defined similarly.

THEOREM I. The operators H_m (as well as the H_m^+) converge to a limit operator H , in the sense that

$$(30) \quad \lim_{m \rightarrow \infty} \int_{\Omega} |Hf - H_m f|^2 d\mu = 0, \text{ for all } f \in L^2(\Omega, \mu),$$

and the operators H, H_m, H_m^+, H_m^- are uniformly bounded in $L^2(\Omega, \mu)$.

PROOF. Accordingly to theorem 2 of the precedent paper [6], if $H_m^+ = T_1 + \dots + T_m$, and the operators T_i satisfy the following properties :

$$(1) \quad (T_i f, g) = (f, T_i g);$$

$$(2) \quad T_i T_j = T_j T_i;$$

$$(3) \quad \|T_i f\|_2 \leq O_2 \cdot \|f\|_2, \|T_i T_j f\|_2 \leq \lambda^{-|i-j|} \|f\|_2;$$

then $\|H_m^+ f\|_2 \leq O_2 \cdot \|f\|_2$. It is easy to see that condition (1) may be replaced by the following two conditions :

$$(2a) \quad T_i T_j^* = T_j^* T_i,$$

$$(3a) \quad \|T_i T_j^* f\|_2 \leq \lambda^{-|i-j|} \|f\|_2.$$

In fact, if T_i satisfy (2), (2a) (3) and (3a), then the operators $2^{-1}\{H_m^+ + (H_m^+)^*\}$ and $(2i)^{-1}\{H_m^+ - (H_m^+)^*\}$ satisfy the conditions (1), (2) and (3) with $\lambda^{1/2}$ instead of λ . Hence by Proposition II, we obtain that the operators H_m^+, H_m^- and H_m are uniformly bounded on $L^2(\Omega, \mu)$. Since $\|H_m f - Hf\|_2 = 0$ for every $f \in \mathbf{D}_2(\Omega)$ and, by lemma 7, $\mathbf{D}_2(\Omega)$ is dense in $L^2(\Omega, \mu)$, it follows that $\|H_m f - Hf\|_2 \rightarrow 0$ for every $f \in L^2(\Omega, \mu)$.

THEOREM II. The operators $\overline{M}^+, \overline{M}^-, \overline{M}$, are dominated by the operators $\overline{H}^+, \overline{H}^-, \overline{H}$ and Λ , as follows :

$$\begin{aligned} \|M^+ h(x)\|^2 &\leq O_2 \{ \|\overline{H}^+ h(x)\|^2 + \Lambda(h^2)(x) + \|\Lambda h(x)\|^2 \\ &\quad + \Lambda(\|\overline{H}^+ h\|^2)(x) \}, \end{aligned} \quad (31)$$

$$\begin{aligned} \|\overline{M} h(x)\|^2 &\leq O_2 \{ \|\overline{H} h(x)\|^2 + \Lambda(h^2)(x) + \|\Lambda h(x)\|^2 \\ &\quad + \Lambda(\|\overline{H} h\|^2)(x) \} \end{aligned} \quad (32)$$

where Λ is the Hardy-Littlewood maximal operator defined by

$$\Lambda h(x) = \sup_{\varepsilon > 0} \{V(\varepsilon)\}^{-1} \cdot \int_{B(x, \varepsilon)} |h(t)| dt.$$

PROOF. Let us fix the number N , and let $m < N$, and let $h(x)$ be a function defined on R^n , $h \in L^2(R^n)$. Consider first the operator $\bar{H}_m^+ = \sum_{i=0}^m \bar{T}_i$. Let x be a fixed point, and x_1 any point of the sphere $B(x, 2^m) = \{|x - x_1| < 2^m\}$. We may assume $x = 0$. Since K_i vanishes for $|t| > 2^i$, we have:

$$\begin{aligned} (33) \quad \bar{H}_N^+ h(x) - \bar{H}_m^+ h(x) &= [\bar{H}_N^+ h(x) - \bar{H}_m^+ h(x_1)] \\ &= \sum_{i=m}^N \int_{R^n} [K_i(x-t) - K_i(x_1-t)] h(t) dt \\ &= \sum_{i=m}^N \int_{R^n} [K_i(x-t) - K_i(x_1-t)] h(t)_{(2^i)} dt, \end{aligned}$$

where $h(t)_{(2^i)} = h(t)$ if $|x - t| < 2 \cdot 2^i$, and zero otherwise. Since $|x - x_1| < 2^m$, by (1), (2), the right hand of (33) is

$$\begin{aligned} (33a) \quad &\leq \sum_{i=m}^N \int_{R^n} \{ |K_i(x-t) - K_i(x_1-t)|^2 dt \}^{1/2} \{ |h(t)_{(2^i)}|^2 dt \}^{1/2} \\ &\leq \sum_{i=m}^N \{ 2^{-ni} \} \int_{R^n} \{ |K_i(x-t) - K_i(x_1-t)|^2 dt \}^{1/2} \int_{|x-t| < 2^{i+1}} |h(t)|^2 dt \}^{1/2} \\ &\leq \sum_{i=m}^N \left(\frac{O_1 2^{m_i}}{2^i} \right)^{1/2} \cdot \frac{O_1}{V(2^i)} \int_{B(2 \cdot 2^i)} |h(t)|^2 dt \}^{1/2} \\ &\leq O_1 \{ \Lambda[|h|^2](x) \}^{1/2}, \end{aligned}$$

where Λ is the Hardy-Littlewood maximal operator.

On the other hand, since $|K_i| \leq 2^{-ni} \cdot O_1$,

$$\begin{aligned} (33b) \quad \bar{H}_m^+ h(x_1) &= \bar{H}_m^+ h_{(2^m)}(x_1) = \\ &= \sum_{i=0}^m \int_{R^n} K_i(x_1-t) h_{(2^m)}(t) dt, \\ \left| \sum_{i=0}^m \int_{R^n} K_i(x-t) h_{(2^m)}(t) dt \right| &\leq \sum_{i=0}^m O_1 \frac{2^{nm}}{2^{ni}} \cdot \frac{1}{V(2 \cdot 2^m)} \int_{B(2 \cdot 2^m)} |h(t)| dt \\ &\leq O_1 \Lambda h(x), \end{aligned}$$

hence

$$|\bar{H}_m^+ h|(x_1) \leq |\bar{H}_N^+ h_{(2^{-m})}(x_1)| + O_2 \Lambda h(x).$$

From the last inequality and (33), (33 a), we obtain that for $|x_1 - x| < 2^m$,

$$\begin{aligned} |\bar{H}_m^+ h(x)|^2 &\leq |\bar{H}_N^+ h(x)|^2 + |\bar{H}_N^+ h(x_1)|^2 + O_1 \Lambda(h^2)(x) \\ &\quad + |\bar{H}_N^+ h_{(2^{-m})}(x_1)|^2 + O_1 |\Lambda h(x)|^2. \end{aligned}$$

Integrating in x_1 over $|x - x_1| < 2^m$ and dividing by $V(2^m)$, we obtain

$$\begin{aligned} (34) \quad |\bar{H}_m^+ h(x)|^2 &\leq |\bar{H}_N^+ h(x)|^2 + O_1 \Lambda(h^2)(x) + O_1 |\Lambda h(x)|^2 \\ &\quad + \frac{1}{V(2^m)} \int_{B(2^m)} |\bar{H}_N^+ h(x_1)|^2 dx_1 + \frac{1}{V(2^m)} \int_{B(2^m)} |\bar{H}_N^+ h_{(2^{-m})}(x_1)|^2 dx_1. \end{aligned}$$

By a), the operators $\bar{H}_N^+$ are uniformly bounded on $L^2(R^n)$, therefore

$$\begin{aligned} &\frac{1}{V(2^m)} \int_{B(2^m)} |\bar{H}_N^+ h_{(2^m)}(x_1)|^2 dx_1 \\ &\leq \frac{1}{V(2^m)} \|\bar{H}_N^+ h_{(2^m)}\|_2^2 \leq \frac{O_2}{V(2^m)} \int_{R^n} |h_{(2^m)}|^2 dt \\ &\leq \frac{O_2}{V(2^m)} \int_{B(2^{m+1})} |h(t)|^2 dt = O_2 \Lambda(h^2)(x). \end{aligned}$$

Hence from (34) we get

$$\begin{aligned} |\bar{H}_m^+ h(x)|^2 &\leq O_2 \{ |\bar{H}_N^+ h(x)|^2 + |\Lambda(h^2)(x)| + \\ &\quad |\Lambda h(x)|^2 + \Lambda(|\bar{H}_N^+ h|^2)(x) \}, \end{aligned}$$

for any $m \leq N$. Since the right hand of the last inequality does not depend on m , it follows that (31) is true, and allowing N go to infinity we obtain (32).

A similar argument may be applied to the operator $\bar{H}_m^-$. In fact, let x be fixed and x_1 vary in the sphere $B(x, 2^{-m}) = \{ |x - x_1| < 2^{-m} \}$.

Then

$$\begin{aligned} (35) \quad \overline{H}_m^- h(x) - \overline{H}_m^- h(x_1) &= \sum_{i=-m}^0 \int_{\mathbb{R}^n} [K_i(x-t) - K_i(x_1-t)] h(t) dt \\ &= \sum_{i=-m}^0 \int_{\mathbb{R}^n} [K_i(x-t) - K_i(x_1-t)] h_{(i)}(t) dt, \end{aligned}$$

where $h_{(i)}(t) = h(t)$ if $|t| < 2 \cdot 2^{-i}$, and zero otherwise.
As in (33 a) we will get:

$$(36) \quad |\overline{H}_m^- h(x) - \overline{H}_m^- h(x_1)| \leq O_1 \{ \Lambda(h^2)(x) \}^{1/2}.$$

On the other hand

$$\begin{aligned} \overline{H}_m^- h(x_1) &= \sum_{i=-m}^0 \int_{\mathbb{R}^n} K_i(x_1-t) h(t) dt \\ &= \overline{H}_N^- h(x_1) - \sum_{i=N}^{-m} \int_{\mathbb{R}^n} K_i(x_1-t) h(t) dt \\ &= \overline{H}_N^- h(x_1) - \sum_{i=N}^{-m} \int_{\mathbb{R}^n} K_i(x_1-t) h(t)_{(m)} dt \\ &= \overline{H}_N^- h(x_1) - \overline{H}_N^- h_{(m)}(x_1) + \sum_{i=-m}^0 \int_{\mathbb{R}^n} K_i(x_1-t) h_{(m)}(t) dt, \end{aligned}$$

and as in (33 b) we will get

$$(36a) \quad |\overline{H}_m^- h(x_1)| \leq |\overline{H}_N^- h(x_1)| + |\overline{H}_N^- h_{(m)}(x_1)| + O_2 \Lambda h(x).$$

From (36) and (36 a) we obtain, as above, that

$$\begin{aligned} (35a) \quad |\overline{H}_m^- h(x)|^2 &\leq O_2 \{ |\overline{H}_m^- h(x)|^2 + |\Lambda(h^2)(x)| \\ &\quad + |\Lambda h(x)|^2 + \Lambda(\overline{H}_m^- h)^2(x) \}, \end{aligned}$$

and this proves the theorem.

THEOREM III. $H_m f(P)$ converges pointwise to Hf on $L^2(\Omega, \mu)$, that is

$$(37) \quad \lim_{m \rightarrow \infty} H_m f(P) = Hf(P),$$

for every $f \in L^2(\Omega, \mu)$, and almost all P . Similarly $H_m^+ f(P)$ converges to $H^+ f(P)$.

PROOF: We shall first prove, that for every $\lambda > 0$, and every $h \in L^2(R^n)$,

$$(38) \quad |E[x: \overline{M}h(x) \geq \lambda]| \leq \frac{O_2}{\lambda^2} \int_{R^n} |h(x)|^2 dx.$$

In fact, by theorem I, $\|\overline{H}h\|_2 \leq O_2 \|h\|_2$, therefore taking in account the theorem 1 of the precedent paper [8],

$$|E[|\overline{H}h|^2 \geq \lambda^2]| \leq \frac{1}{\lambda^2} \int_{R^n} |\overline{H}h|^2 dt \leq \frac{O_2}{\lambda^2} \int_{R^n} |h(t)|^2 dt,$$

$$|E[\Lambda(h^2)(x) \geq \lambda^2]| \leq \frac{1}{\lambda^2} \int_{R^n} |h(t)|^2 dt,$$

$$|E[|\Lambda h|^2(x) \geq \lambda^2]| \leq \frac{1}{\lambda^2} \int_{R^n} |\Lambda h|^2 dt \leq \frac{O_2}{\lambda^2} \int_{R^n} |h(t)|^2 dt,$$

$$|E[\Lambda(\overline{H}h)^2(x) \geq \lambda^2]| \leq \frac{O_2}{\lambda^2} \int_{R^n} |\overline{H}h(t)|^2 dt \leq \frac{O_2}{\lambda^2} \int_{R^n} |h(t)|^2 dt.$$

Since, by theorem II,

$$\begin{aligned} |E[\overline{M}h \leq \lambda]| &\geq |E[|\overline{H}h|^2 \geq \lambda^2/4]| + |E[\Lambda(h^2) \geq \lambda^2/4]| \\ &\quad + |E[|\Lambda h|^2 \geq \lambda^2/4]| + |E[\Lambda(\overline{H}h)^2 \geq \lambda^2]|, \end{aligned}$$

we obtain (38).

Now we shall prove that, for every $\lambda > 0$ and every $f \in L^2(\Omega, \mu)$,

$$(39) \quad \mu(E[P; Mf(P) \geq \lambda]) \leq \frac{O_2}{\lambda^2} \int_{\Omega} |f(P)|^2 d\mu(P).$$

If fact, let N be a fixed number, $E(t, N)$ the set of all the points $P \in \Omega$ such that $M_N f(P, t) \geq \lambda$, and $E(P, N)$ the set of all the points $x \in R^n$ such that $|x| \leq 2^N$ and $M_N f(P, x) \geq \lambda$.

Then

$$\begin{aligned} \mu\{E[M_N f(P) \geq \lambda]\} &= \mu\{E(O, N)\} \\ &= \frac{1}{V(2^N)} \int_{E(2^N)} \{\mu E(t, N)\} dt = \frac{1}{V(2^N)} \int_{\Omega} |E(P, N)| d\mu(P). \end{aligned}$$

It is clear that

$$E(P, N) = E[\bar{M}_N f(P, \cdot)(x) \geq \lambda] \cap B(2^N).$$

Taking in account that, by lemma 1,

$$\bar{H}_m f(P, \cdot)(x) = \bar{H}_m [f(P, \cdot)_{(2 \cdot 2^N)}](x),$$

for $|x| \leq 2^N$ and $m < N$, we will have, by (38),

$$\begin{aligned} |E(P, N)| &= |E[x; \bar{M}_N f(P, \cdot)(x) \geq \lambda] \cap B(2^N)| \\ &= |E[x; \bar{M}_N |f(P, \cdot)_{(2 \cdot 2^N)}](x) \geq \lambda]| \\ &\leq \frac{O_2}{\lambda^2} \int_{\mathbb{R}^n} |f(P, t)_{(2 \cdot 2^N)}|^2 dt \leq \frac{O_2}{\lambda^2} \int_{B(2 \cdot 2^N)} |f(P, t)|^2 dt \end{aligned}$$

Hence

$$\begin{aligned} \mu\{E[M_N f(P) \geq \lambda]\} &= \frac{1}{V(2^N)} \int_{\Omega} |E(P, N)| d\mu \\ &\leq \frac{1}{V(2^N)} \cdot \frac{O_2}{\lambda^2} \int_{\Omega} \left\{ \int_{B(2 \cdot 2^N)} |f(P, t)|^2 dt \right\} d\mu \\ &= \frac{V(2 \cdot 2^N)}{V(2^N)} \cdot \frac{O_2}{\lambda^2} \int_{\Omega} |f(P)|^2 d\mu = \frac{O_2}{\lambda^2} \int_{\Omega} |f|^2 d\mu. \end{aligned}$$

Allowing N to go to infinity we obtain (39).

From (39), Proposition I, b), and Proposition III, we obtain that (37) holds for every $f \in L^2(\Omega, \mu)$, and this proves the theorem.

Applications. Theorem I contains as special cases, the mean ergodic theorem of v. Neumann, the Lusin-Riesz theorem for the ordinary Hilbert transform, and the Calderón-Zygmund-Mjlin theorem for the n -dimensional Hilbert transforms. Similarly, theorem II contains as special cases (for the space L^2) the individual ergodic theorem, the Lusin theorem for the ordinary Hilbert transform and the Calderón-Zygmund theorem for the n -dimensional Hilbert transforms.

In fact, let $\Omega = \mathbb{R}^1 \setminus \{x\}$ = the real line, $P = x$, μ = the ordinary Lebesgue measure, $\sigma_t P = \sigma_t x = x + t$, and let $K(x) = x^{-1}$ if

$1 \leq |x| < 2$, and $K(x) = 0$ otherwise. It is clear that $K(x)$ so defined, fulfils the conditions $\alpha) - \delta)$, and in this case H_m is given by

$$H_m f(x) = \sum_{i=-m}^m \int_{2^i < |x-t| < 2^{i+1}} \frac{f(t)}{x-t} dt = \int_{2^{-m} < |x-t| < 2^m} \frac{f(x)}{x-t} dt$$

Therefore theorems I and II furnish the mean and pointwise convergence of the Hilbert transform $H_\varepsilon f(x)$ for the case where $\varepsilon = 2^{-m}$, $m = 1, 2, \dots$. However it is easy to reduce the convergence of $H_\varepsilon f$, for any $\varepsilon > 0$, to that of $H_\varepsilon f$ for $\varepsilon = 2^{-m}$. Similarly, let $\Omega = \mathbb{R}^2 = \{z\}$, $\mu =$ the ordinary Lebesgue measure in $\mathbb{R}^2$, $\sigma_u P = P + u = z + u$ and let $K(z)$ be defined by

$K(z) = \omega(\theta)/|z|^2$, ($z = |z|e^{i\theta}$), if $1 \leq |z| < 2$, and $K(z) = 0$ otherwise, where the function $\omega(\theta)$ satisfies the two following conditions:

$$\int_0^{2\pi} \omega(\theta) d\theta = 0$$

$$\int_0^{2\pi} |\omega[|\theta - d(\theta)| - \omega(\theta)]| d\theta \leq O_1 \cdot d$$

if $|d(\theta)| \leq d$ (cfr. [6]). In this case we obtain that $H_m f(z) = H_\varepsilon f(z)$, $\varepsilon = 2^{-m}$, are the 2-dimensional Hilbert transforms given by (I'), and similarly for the n -dimensional transforms in $\mathbb{R}^n$, $n \geq 2$. Finally, let $\Omega = \{P\}$ be a general measure space with a measure μ and a n -parameter group of measure perserving transformations σ_x , $x \in \mathbb{R}^n$, and let $K(x)$ be defined by: $K(x) = +1$ if $1 \leq |x| \leq 2$, $K(x) = -1$ if $|x| < 1$, $K(x) = 0$ otherwise. It is easily verified that $K(x)$ so defined satisfies conditions $\alpha) - \delta)$, and that, in this case,

$$H_m^+ f = D_{2m} f - D_1 f,$$

where $D_N f$ are the ergodic operators defined by (in the 1-dimensional case)

$$D_N f(P) = \frac{1}{N} \int_0^N f(\sigma_t P) dt.$$

Since the change of $D_N f$ by $D_N f - D_1 f$ is not essential, theorems I and III furnish, in this case, the corresponding ergodic theorems. Similarly $H_m^- f$ reduces, in this case, to the differential operators $D_\varepsilon f(x)$.

Thus, both, the ergodic theory and theory of Hilbert transforms (and in particular the theory of differential operators $Dz f$), are special cases of the general theory of operators H_m, H_m^+ , defined by (10a). The ergodic theory corresponds to the case $K(x) = 1$ if $1 \leq |x| < 2$, $K(x) = -1$ if $|x| < 1$ and $K(x) = 0$ otherwise. The theory of Hilbert transforms corresponds to the case $K(x) = w(x)/|x|^n$ if $1 \leq |x| \leq 2$, $K(x) = 0$ otherwise, and $\Omega = R^n$. If in the last case we take a general space $\Omega \neq R^n$, then we obtain an « ergodic » generalization of the Hilbert transforms.

4. Generalization to locally compact abelian groups. We shall show now that, using the Wiener-Beurling-Kaplansky tauberian theorem, the theorems I-III are easily extended to general groups G , that is to kernels $K(x)$ defined on a general group G , instead of R^n .

In order to make clear the generalization to general groups, we first observe that the kernels $K_i(x)$ defined by (7) possess the following important property.

Consider first the 1-dimensional space $R^1 = \{x\}$, and let $K_i(x) = 2^{-i}K(2^{-i}x)$, where $K(x)$ satisfies conditions $\alpha) - \delta)$. Conditions $\alpha)$ and $\delta)$ imply that the Fourier transform of $K_i(x)$,

$$\hat{K}_i(u) = \int_{-\infty}^{\infty} K_i(t) e^{-iut} dt,$$

is an analytic function in u , for $-\infty < u < \infty$, and condition $\beta)$ implies that $\hat{K}_i(0) = 0$ for all $i = \pm 1, \pm 2, \dots$. Hence, it is easy to see that the kernels $K(x)$ satisfy following condition:

$\varepsilon)$ $u_0 = 0$ is the only point, such that $\hat{K}_i(u_0) = 0$ for all $i = \pm 1, \pm 2, \dots$ (unless $K = 0$).

In fact, from $K_i(x) = 2^{-i}K(2^{-i}x)$ it follows that $\hat{K}_i(u) = \hat{K}(2^i u)$, therefore $\hat{K}_i(u_0) = 0$, for all i and for $u_0 \neq 0$, implies $\hat{K}(2^{-i}u_0) = 0$ for $i = 1, 2, \dots$, and $\hat{K}(u)$ being an analytic function, this gives $\hat{K} \equiv 0$, $K \equiv 0$.

Now, if the kernels $K(x)$ and $K_i(x)$ are defined in the n -dimensional space R^n , $n > 1$, then $\hat{K}_i(u_0) = 0$, for all i , still implies that $\hat{K}(u) = 0$ on the whole line joining O with u_0 , but we cannot assure $u_0 = 0$ is the only point at which all the $\hat{K}_i$ vanish. However, we can assure that the sequence K_i is a linear combination of se-

quences satisfying this property. More precisely: there exist two kernels $K'(x)$ and $K''(x)$, satisfying conditions $\alpha) - \delta)$ and condition $\varepsilon)$, and such that $K(x) = K'(x) + K''(x)$ and $K_i(x) = K'_i(x) + K''_i(x)$. In fact, let $x = (x_1, \dots, x_n)$ and let $K'(x) = O_1 \{ \varphi_1(x) - \varphi_2(x) \}$, where

$\varphi_1(x) = 1$ in the cube $|x_1| < \pi, \dots, |x_n| < \pi$, and $\varphi_1(x) = 0$ otherwise,

$\varphi_2(x) = 1$ in the cube $|x_1| < \frac{\pi}{2}, \dots, |x_n| < \frac{\pi}{2}$, and $\varphi_2(x) = 0$ otherwise.

Then $\hat{K}' \neq 0$ on the sphere $|u| = 1$, and choosing the constant O_1 sufficiently large we will have also $\hat{K}''(u) \neq 0$ on $|u| = 0$, where $K'' = K - K'$. From here, it follows that K' and K'' satisfy condition $\varepsilon)$ because $\hat{K}'_i(u_0) = 0$ for all i (or $\hat{K}''_i(u_0) = 0$ for all i) would imply $\hat{K}'(u) = 0$ for all u of the line Ou_0 , and hence for some u of the sphere $|u| = 1$.

Consider now a locally compact abelian group $G = \{x\}$ (we shall write additively, $x + y$, the group operation in G) and let dx denote the Haar measure of G . Let $K_i(x)$ be a sequence of functions defined in G with the following properties:

a) $K_i(x) = 0$ outside of a compact neighborhood U_i of the origin, and $2U_i = U_i + U_i \subset U_{i+1}$, for all i .

b) $\int_G K_i(x) dx = 0$, for all i .

c) $\|K_i * K_j\| \leq 2^{-|i-j|}$ for all $i, j = 1, 2, \dots$

where

$$f * g(x) = \int_G f(t) g(t-x) dt$$

denotes the convolution product of f and g .

d) $|K_i(x)| \leq O_1(|U_i|)^{-1}$, for all i , and for all $x \in G$.

e) $\hat{K}_i(u_0) = 0$ for all $i = 1, 2$, implies $u_0 = 0$,

where $u_0 \in \hat{G}$ = the dual group of G , and

$$\hat{f}(u) = \int_G f(x) (x, u) dx$$

denotes the Fourier transform of f .

f) For each set U_i and each integer n , there is a compact set $W_i \supset nU_i$ such that $|W_i + nU_i| \leq 0 \cdot |nU_i|$, where 0 is an absolute constant. Here nU denotes $U + U + \dots + U$, and $|U|$ = measure of the set U .

Obviously, the results given below, will apply to sequences $\{\lambda_i' K_i' + \lambda_i'' K_i''\}$ which are linear combinations of sequences satisfying the conditions a)-f).

Let $\Omega = \{P\}$ be a measure space with a measure μ , and let $\{\sigma_x\}$, $x \in G$, be a measurable group of measure perserving transformations of Ω , so that $\sigma_{x+y}P = \sigma_x(\sigma_yP)$, $\sigma_0P = P$. To any function $f(P)$ defined on Ω we assign the function $f(P, t) = f(\sigma_tP)$ defined on $\Omega \times G$, and we define, as above, the operators

$$(10'') \quad \bar{T}_i f(P) = \int_G f(P, t) K_i(t) dt = \int_{G_i} f(P, t) K_i(t) dt,$$

$$(10a'') \quad H_m = \sum_i \bar{T}_i,$$

$$(11'') \quad \bar{T}_i g(x) = \int_G g(x+t) K_i(t) dt, \quad (x \in G),$$

$$(11a'') \quad \bar{H}_m = \sum_i \bar{T}_i,$$

where $\bar{H}_m$ acts on the group G , and H_m on the space Ω .

If U is an open relatively compact set of G , and $g(x)$ is defined on G , we denote by $g_{(U)}$ the function equal to $g(x)$ if $x \in U$ and zero otherwise, and similarly for $f(P, \cdot)_{(U)}$. $\varphi(x) = \varphi(x, U)$ will denote a function such that $\varphi(x) = \varphi(x)_{(U)}$, and $|U|$ will denote the measure of U . Finally φ_e will denote the function defined on G by

$$(12'') \quad \varphi_V(x) = |V|^{-1} \quad \text{if } x \in V, \text{ and zero otherwise,}$$

where $V = V(z)$ is a relatively compact open neighborhood of certain point $z \in G$.

We shall first generalize the mean theorem I. For this purpose let us observe that the assumption f) permits to extend the proof of lemma 4 to our general group G , only that the set $|x| > N$ has to be replaced now by an open relatively compact neighborhood U of the origin, such that $U \subset nU_i$ for some n .

It is well known that lemma 5 is true for general groups G (the lemma is immediate for continuous functions h vanishing out of a compact set, and by Young's inequality it is easily extended to

functions $h \in L^p$). Similarly, using f' , it is easy to see that lemma 6 also holds in our case (of course φ_ε has to be replaced by φ_V with $V \subset U_i$, and the V forming a directed fundamental set of neighborhoods of the identity). Hence Lemma 7 is true for the group G .

The proof of the Proposition III does not apply to general groups G , because it uses specific properties of the euclidean space. However, using the condition c) and the tauberian theorem for topological groups, Proposition III can be proved as follows.

PROPOSITION III^a. *There is a set $\mathbf{D}(\Omega, G)$ of functions dense in $L^2(\Omega, \mu)$, and such that for each function $f \in \mathbf{D}(\Omega, G)$ the sequence $H_m f$ converges in the mean and pointwise to a limit Hf .*

PROOF. Let $\mathbf{D}'(\Omega, G)$ be the set of all functions $f \in L^2(\Omega, \mu)$ of the form

$$f = g(P, \cdot) * \varphi_V(u) - g(P, \cdot) * \varphi_V(u') = \int g(P, t) [\varphi_V(u+t) - \varphi_V(u'+t)] dt,$$

where $g \in L^2(\Omega, \mu) \cap L^1(\Omega, \mu)$, and $u + V \subset nV_i$, $u' + V \subset nV_i$, for some n and some i . If $\mathbf{D}''$ is the orthogonal set to $\mathbf{D}'(\Omega, G)$, then $\mathbf{D}(\Omega, G) = \mathbf{D}'(\Omega, G) + \mathbf{D}''(\Omega, G)$ is dense in L^2 , and is enough to prove the theorem for a set dense in $\mathbf{D}(\Omega, G)$. As above we will have that if $h \in \mathbf{D}''$ then $h(P) = h(\sigma_x P)$ for every x such that $x \in nU_i$ for some n and some U_i and by b) we will have $H_m h(P) = 0$, so that the proposition is true for $\mathbf{D}''(\Omega, G)$.

Let $f \in \mathbf{D}'(\Omega, G)$, $f = g * \varphi_V(u) - g * \varphi_V(u')$, with $u + V \subset nU$, $u' + V \subset nU'$.

By b) and e), $u = 0$ is the only point at which all the Fourier transforms $K_i(u)$ vanish. Hence, by a tauberian theorem of Beurling and Kaplansky ([16], [17]), any function $\varphi(x)$, defined on G and such that $\int \varphi(x) dx = 0$, can be approximated in $L^1(G)$ by linear combinations of the form: $\sum_{i=1}^m c_i K_i(x - x_i)$. That is, given $\varepsilon > 0$, there exist m constants $c_1, \dots, c_m$, and m points $x_1, \dots, x_m \in G$, such that

$$\left\| \varphi - \sum_{i=1}^m c_i K_i(x - x_i) \right\|_1 < \varepsilon.$$

We take $\varphi = \varphi_V - \varphi_{V'}$. Then, by lemma 4,

$$\|f' - g * \psi\|_p = \|g * (\varphi - \psi)\|_p \leq \|g\|_p \cdot \|\varphi - \psi\|_1 \leq \varepsilon \cdot \|g\|_p,$$

$p = 1, 2$, where $\psi = \sum c_i K_i(x - x_i)$ and $g * \psi(P) = \int_G g(\sigma_x P) \psi(x) dx$.

Hence the functions of the form $h = g \star \psi$ are dense in $\mathbf{D}'(\Omega, G)$, and it is sufficient to prove the theorem for these functions h .

For such a function $h = g \star \psi$, $\psi = \sum_{i=1}^m c_i K_i(x - x_i)$, we have by lemma 4

$$\|(H_{N+p}h - H_N h)\|_2 \leq \|g\|_2 \cdot \left\| \sum_{j=N}^{N+p} \psi \star K_j \right\|_1 = \|g\|_2 \cdot \left\| \sum_{j=N}^{N+p} \sum_{i=1}^m c_i K_i \star K_j \right\|_1,$$

and by condition c), we obtain

$$\|(H_{N+p}h - H_N h)\|_2 \leq (c_1 + \dots + c_m) \sum_{i=N-m}^{\infty} 2^{-i} \cdot \|g\|_2 = 0$$

for $N \rightarrow \infty$. Hence $H_m h$ converge in the mean to a limit Hh .

Similarly we will have that, since $g \in L^2 \cap L_1$,

$$\sum_{i=1}^{\infty} \int |T_i h(P)| d\mu \leq \sum_{i=1}^{\infty} \|T_i \psi\|_1 \cdot \|g\|_1 = \|g\|_1 \cdot \sum_{i=1}^{\infty} \|\psi \star K_i\|_1 \\ \|g\|_1 \cdot (c_1 + \dots + c_m) \sum_{i=0}^{\infty} 2^{-i} < \infty,$$

and this implies the pointwise convergence of $H_m h(P)$, almost everywhere. This proves Proposition III^a.

From Proposition III^a, condition c) and theorem 2 of the precedent paper [6] we obtain immediately:

THEOREM I^a. *The operators $H_m f$ converge to Hf , and are uniformly bounded, on $L^2(\Omega, \mu)$. Moreover, $H_m f(P)$ converges pointwise to $Hf(P)$, almost everywhere, for all f of a dense set in $L^2(\Omega, \mu)$.*

REMARK. In the proof of theorem I^a, condition d) has not been used, and conditions a) and f) have been used only in the proof of Lemma 4.

If $\Omega = G$, and $\sigma_t P = \sigma_t x = x + t$, then Lemma 4 is unnecessary and theorem I^a is true if the conditions b), c) and e) alone are satisfied. Thus:

THEOREM 1b. *If the kernels $K_i(x)$ defined on G , satisfy conditions b), c) and e), then operators $\bar{H}_m f$ converge to a limit operator $\bar{H}f$, and are uniformly bounded, on $L^2(G)$. Moreover, $\bar{H}_m f(x)$ is pointwise convergent to $\bar{H}f(x)$ almost everywhere, for all f of a dense set in $L^2(G)$.*

Since theorem I^a applies to sequences K_i which are linear combinations of sequences satisfying a)-f), it is clear that theorem 1 is a

special case of theorem I^a. In particular, theorem I^a gives a generalization of v. Neuman's ergodic theorem to general groups G .

We pass now to the generalization of theorems II and III. The above proofs of these theorems are based on the use of the Hardy-Littlewood maximal theorem. However, the maximal theorem is used there only for the sets U_i . More precisely, let $\Delta f(x)$, $x \in G$, be defined by :

$$(40) \quad \Delta f(x) = \sup_i \frac{1}{|U_i|} \int_{U_i} f(x+t) dt,$$

where $|U_i|$ denotes the measure of the set U_i , and U_i are the sets of condition a).

Then for the proofs of theorems II and III it is sufficient to know that this special operator Δf satisfies the Hardy-Littlewood maximal theorems, that is that Δ is of the type p , for $p > 1$, and satisfies condition (1)

A sufficient condition for the sets U_i , under which the corresponding operator Δf satisfies the maximal theorem, has been given by A. Calderón [9]. Calderón showed that the Hardy-Littlewood maximal theorem is true for Δf if Δ is defined by (40) and if the open sets U_i satisfy the following condition :

f_1) The measure of U_{2i} is $\leq \alpha \cdot (\text{measure of } U_i)$, for all i where α is a constant independent of i .

Observe that condition f_1) is considerably stronger than condition f). It is also necessary to observe that Calderon assumes the space to be of finite measure. However his proof, in what the Hardy-Littlewood theorem concerns, is easily extended to the general case by using an argument of E. Hopf [18]. [In fact, in the proof of the dominated theorem of Calderon's paper, it is enough to define the function $G_m(x)$, in the notations of Calderon's paper, as follows: $G_m(x) = [F(x) - \lambda] \varphi_E$, where E is any set of finite measure such that $E \subset D_m$, and $E \supset$ the set of points x for which $F(x) \geq \lambda$. Applying the theorem with $\lambda = 0$ and letting E tend to D_m , it follows the desired result]. On the other hand it is clear that we may limit the function Δf only to the points x belonging to the set $\cup_i (U_i + U_i)$.

Replacing condition f) by the stronger condition f_1), the above proofs of theorems II and III apply to the present case of general groups G and we obtain :

THEOREM II^a. *If the kernels $\overline{K}_i$ satisfy conditions a), b), c), d), e) and f_1) then the operator $\overline{M}h(x) = \sup \overline{H}_m h(x)$ is dominated by the*

operators $\overline{Hh}(x)$ and $\Lambda h(x)$ by the formula (32), where Λ is defined by (40)

THEOREM III^a. *If the kernels K_i satisfy condition a), b), c), d), e) and f_1), then $H_m f(P) \rightarrow Hf(P)$, almost everywhere, for every $f \in L^2(\Omega, \mu)$. Similarly can be defined the sequence K_{-i} for negative i and theorems I^a, II^a and III^a hold also for these sequences.*

Thus, these theorems contain as special cases the theorems II and III as well as Calderon's generalization [9] of the ergodic theorem to general groups G .

The above theorems furnish an unification of the «discrete» and «continuous» theories.

Generally the ergodic theorems are proved separately for the discrete group $\{\sigma^n\}$, $n = 0, \pm 1, \pm 2, \dots$ and for the continuous group of the reals $\{\sigma_t\}$, $-\infty < t < \infty$. Likewise are treated separately the ordinary Hilbert transforms and the «discrete» Hilbert transforms for sequences introduced by M. Riesz (cfr. [2]). However all these theories are unified by the theory of the operators H_m in general groups G which apply to both, the continuous and the discrete cases.

5. Mean and pointwise convergence in L^p . — We pass now to consider the spaces L_p , $p \neq 2$, and prove the Riesz inequality, and the pointwise convergence, for these spaces. For the sake of simplicity, we shall consider the kernels $K_i(x)$ defined in the euclidean space $R^n = \{x\}$ by (7), where $K(x)$ is a kernel satisfying $\alpha) - \delta)$. By the argument used in § 4 it will be clear that the following results can be extended to general groups G and kernels K_i satisfying $a) - f)$.

THEOREM IV. *Let T_i, H_m be the operators defined by (10), (10a), where K_i are the kernels defined by (7). Then the operators H_m are uniformly bounded in the space L^p , if $p > 1$, that is,*

$$\int_{\Omega} |H_m f(P)|^p d\mu \leq O_p \int_{\Omega} |f|^p d\mu,$$

for every $f \in L^p$.

Moreover the operators H and H_m satisfy uniformly condition (1), that is

$$\int_E |H_m f(P)|^q d\mu \leq O_\alpha (\mu E)^{1-\alpha} \int_{\Omega} |f|^q d\mu, \quad \text{if } 0 < \alpha < 1.$$

PROOF. Let $h(x)$ be a step-function defined on $R^n = \{x\}$, and let $B = B(c, r)$ be a sphere, with center at c and radius r , such that $h(x) = 0$ if $x \in R^n - B$. Let $\gamma(x)$ be defined by

$$\gamma(x) = \begin{cases} \frac{1}{c(B)} \int_B h(t) dt, & \text{if } x \in B, \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\int_{R^n} \gamma(x) dx = \int_{R^n} h(x) dx, \quad \|\gamma\|_1 \leq \|h\|_1.$$

Then the function $\varphi(x) = h(x) - \gamma(x)$ satisfies the conditions

$$\int_{R^n} \varphi(x) dx = 0, \quad \varphi(x) = 0 \quad \text{if } x \in R^n - B.$$

We may assume that $c = 0$, and let s be an integer such that $2^{s-1} < r \leq 2^s$ (r = the radius of B). If x_0 is a point such that $|x_0 - c| = |x_0| > 2^{s+1} \geq 2r$, since $K_i(t) = 0$ for $|t| > 2^i$ and $\varphi(x) = 0$ for $x \in R^n - B$ it is clear that $\varphi \star K_i(x_0) = 0$ for any $i \leq s$. Thus,

$$\overline{H_m} \varphi(x_0) = \sum_{i=1}^m h \star K_i(x_0) = \sum_{i=s}^m h \star K_i(x_0),$$

for any x_0 such that $|x_0| > 2^{s+1} \geq 2r$. Therefore by lemma 2 we obtain:

$$\begin{aligned} \int_{|x| > 2r} |\overline{H_m} \varphi(x)| dx &= \int_{|x| > 2r} \left| \sum_{i=s}^m \varphi \star K_i(x) \right| dx \\ &\leq \sum_{i=s}^m \|\varphi \star K_i\|_1 \leq O_1 2^{s+1} \left(\frac{1}{2^s} + \frac{1}{2^{s+1}} + \dots \right) \cdot \|\varphi\|_1 \\ &\leq 2 O_1 (\|h\|_1 + \|\gamma\|_1) = O_1' \|h\|_1. \end{aligned}$$

This shows that the operators $\overline{H_m}$ satisfy uniformly, with the same constant O_1 , condition (p) of the Corollary 1 of the precedent paper [7]. Since the operators $\overline{H_m}$ are uniformly bounded on $L^2(R^n)$ (by theorem I), we obtain by the Corollary 1 of the precedent paper, that

$$(41) \quad \int_{R^n} |\overline{H_m} h(x)|^p dx \leq O_p \int_{R^n} |h|^p dx, \quad \text{if } p > 1,$$

$$(42) \quad \int |\overline{H}_m h(x)|^p dx \leq O_p [E[1-\alpha]] \int |h| dx, \quad \text{if } 0 < \alpha < 1,$$

where O_p and O_α depend only on p and α , respectively.

[Corollary 1 of the precedent paper gives (41) with $p < 2$. However, the arguments used in Proposition 4 of the precedent paper shows that (41) is true for any $p > 2$].

By lemma 1,

$$H_m f(P, x) = \overline{H}_m [f(P, \cdot)_{(2 \cdot 2^m)}](x)$$

for $|x| < 2^m$, hence using (41) and the measure perserving property of σ_x , we obtain (denoting the volume of $B(2^m)$ by $V(m)$):

$$\begin{aligned} \int_{\Omega} |H_m f(P)|^p d\mu &= \frac{1}{V(m)} \int_{B(2^m)} \int_{\Omega} |H_m f(P, x)|^p d\mu dx \\ &= \frac{1}{V(m)} \int_{\Omega} \int_{B(2^m)} |\overline{H}_m [f(P, \cdot)_{(2 \cdot 2^m)}](x)|^p dx d\mu \\ &\leq \frac{O_p}{V(m)} \int_{\Omega} \int_{R^n} |f(P, \cdot)_{(2 \cdot 2^m)}|^p dx d\mu = \\ &\quad \frac{O_p}{V(m)} \int_{\Omega} \int_{B(2 \cdot 2^m)} |f(P, x)|^p dx d\mu \\ &= O_p \cdot \frac{V(m+1)}{V(m)} \int |f(P)|^p d\mu = O_p' \int |f(P)|^p d\mu. \end{aligned}$$

Thus, the operators H_m are uniformly bounded in $L^p(\Omega, \mu)$, if $p > 1$. Since (42) is equivalent to

$$[E|x|; |\overline{H}_m h(x)| \geq \lambda] \leq \frac{O_1}{\lambda} \int_{R^n} |h(x)| dx,$$

by the same argument as the used at the end of the proof of theorem IV, we obtain

$$\mu\{E[P; |H_m f(P, \cdot)| \geq \lambda]\} \leq \frac{O_1}{\lambda} \int_{\Omega} |f(P)| d\mu,$$

and this proves the theorem.

In proving the maximal theorems for the operators H_m we shall assume, instead of γ_1 , the following stronger condition:

γ_1) If $K'(x)$ is the kernel defined by

$K'(x) = 2^n K(2x) + K(x) + 2^{-2} K(2^{-1}x)$ (n -dimensional of R^n)

then

$$(43) \quad |K'(\xi_1) - K'(\xi_2)| \leq O_1 |\xi_1 - \xi_2|,$$

if $|\xi_i| \leq 2^{-1}$, $i=1, 2$, or if $2^{-1} - |\xi_i| \leq 2$, $i=1, 2$.

It is easy verified that condition γ_1 is fulfilled in the case of the Hilbert transforms in R^n as well in the case of the ergodic operators. It is obvious that the sets $|x| \leq 2^{-1}$ and $2^{-1} - |x| \leq 2$, in condition γ_1 , may be replaced by an other partition of the sphere $|x| \leq 2$ in two similar sets.

Since $K_i(x) = 2^{-ni} K(2^{-i}x)$, (43) implies that

$$(44) \quad |K'_i(x) - K'_i(x')| \leq \frac{O_1 |x - x'|}{2^{ni} \cdot 2^i}$$

if $2^{n-1} \leq |x|, |x'| \leq 2^{n+2}$, or if $|x|, |x'| \leq 2^{n-1}$. Here $K'_i(x)$ denotes the kernel $K'_i(x) = 2^{-ni} K'(2^{-i}x)$.

THEOREM V. Let $K(x)$ be a kernel satisfying conditions α , β , γ_1 and δ . Then the maximal operator Mf , defined by (28a), is of the type p , for each $p > 1$, and satisfies condition (I). Moreover, the operators $\bar{H}_m h(x)$ possess the following property: for each m , for each step function $h(x)$, for each $x_0 \in R^n$, and for each x_1 such that $|x - x_1| < 2^m$, it is true that

$$(45) \quad |\bar{H}_m h(x_0)| \leq O_1 \{ |\bar{H}h(x)| + |\bar{H}[h(2^m)](x_1)| + |\bar{H}h(x_1)| + \Lambda h(x) \},$$

where $\bar{H}$ is the limit operator given by theorem I and III, and Λ is the Hardy-Littlewood maximal operator

PROOF. Let $x \in R^n$ be a fixed point and x' such that $|x - x'| < 2^m$, $m \leq N$. As in (33) of the proof of theorem II, we shall have

$$(46) \quad \begin{aligned} \bar{H}_{N^+} h(x) - \bar{H}_m^+ h(x) &= [\bar{H}_{N^+} h(x_1) - \bar{H}_m^+ h(x_1)] \\ &= \sum_{i=m}^N \int_0^1 [K_i(x-t) - K_i(x_1-t)] h(t) dt \\ &= \int_0^1 \sum_{i=m}^N [K_i(x-t) - K_i(x_1-t)] h(t) dt \\ &= \sum_{\alpha=1}^{\infty} J_{\alpha}. \end{aligned}$$

where

$$J_s = \int_{2^s < |x-t| < 2^{s+1}} \sum_{i=m}^N [K_i(x-t) - K_i(x_1-t)] h(t)_{(2^i)} dt.$$

For a fixed $s > m$, and for $2^s < |x-t| < 2^{s+1}$, we will have, taking in account (44) and α_1 ,

$$\begin{aligned} & \left| \sum_{i=m}^N [K_i(x-t) - K_i(x_1-t)] h(t)_{(2^i)} \right| \\ &= \left| \sum_{i=s-1}^N [K_i(x-t) - K_i(x_1-t)] h(t)_{(2^i)} \right| \\ &\leq \left| \sum_{i=s-1}^{s+1} \right| + \left| \sum_{i=s+2}^N \right| \\ &\leq |K_{s'}'(x-t) - K_{s'}'(x_1-t)| |h(t)_{(2^i)}| \\ &\quad + \sum_{i=s+2}^{N-1} |K_i'(x-t) - K_i'(x_1-t)| |h(t)_{(2^i)}| \\ &\leq \sum_{i=s}^{N-1} \frac{O_1 |x-x_1|}{2^{ni} \cdot 2^i} |h(t)_{(2^i)}| \leq \sum_{i=s}^{\infty} \frac{O_1 \cdot 2^m}{2^i} \cdot \frac{1}{2^{ni}} |h(t)_{(2^i)}|. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{s=m}^{\infty} |J_s| &\leq \sum_{s=m}^{\infty} \sum_{i=s}^{\infty} \frac{O_1 \cdot 2^m}{2^{ni}} \frac{1}{2^i} \int_{|x-t| < 2^i} |h(t)| dt \\ &\leq \sum_{s=m}^{\infty} \sum_{i=s}^{\infty} \frac{O_1 \cdot 2^m}{2^i} \Delta h(x) \leq O_1 \Delta h(x). \end{aligned}$$

On the other hand, by β_1 ,

$$\begin{aligned} \left| \sum_{s=1}^m J_s \right| &\leq \sum_{i=m}^N \int_{|x-t| < 2^m} [K_i(x-t) - K_i(x_1-t)] |h(t)| dt \\ &\leq \sum_{i=m}^N \frac{O_1}{2^{ni}} \int_{|x-t| < 2^m} |h(t)| dt = \sum_{i=m}^N \frac{O_1}{2^{ni}} \cdot 2^{nm} \frac{1}{2^{nm}} \int_{|x-t| < 2^m} |h(t)| dt \end{aligned}$$

$$= \sum_{i=m}^N O_1' \cdot \frac{2^{nm}}{2^{ni}} \Delta h(x) = O_1' \Delta h(x).$$

Thus, we obtain now instead of (33a) the following inequality:

$$(47) \quad \left| \overline{H}_N^{+h}(x) - \overline{H}_m^{+h}(x) - [\overline{H}_N^{+h}(x_1) - \overline{H}_m^{+h}(x_1)] \right| \leq O_1 \Delta h(x).$$

From (47) and (33b) of the proof of theorem 11, we get

$$(48) \quad \left| \overline{H}_m^{+h}(x) \right| \leq O_1 \{ \left| \overline{H}_N^{+h}(x) \right| + \left| \overline{H}_N^{+h}(x_1) \right| + \left| \overline{H}_N^{+h}(\varrho_m)(x_1) \right| + \Delta h(x) \}.$$

Similarly, using (35) and (36a), we will deduce that

$$(48a) \quad \left| \overline{H}_m^{-h}(x) \right| \leq O_1 \{ \left| \overline{H}_N^{-h}(x) \right| + \left| \overline{H}_N^{-h}(x_1) \right| + \left| \overline{H}_N^{-h}(\varrho_m)(x_1) \right| + \Delta h(x) \},$$

and therefore

$$(48b) \quad \left| \overline{H}_m^h(x) \right| \leq O_1 \{ \left| \overline{H}_N^h(x) \right| + \left| \overline{H}_N^h(x_1) \right| + \left| \overline{H}_N^h(\varrho_m)(x_1) \right| + \Delta h(x) \},$$

for any $m \leq N$. Since the right hand of the last inequality does not depend on m it follows that

$$(49) \quad \left| \overline{M}_N^h(x) \right| \leq O_1 \{ \left| \overline{H}_N^h(x) \right| + \left| \overline{H}_N^h(x_1) \right| + \left| \overline{H}_N^h(\varrho_m)(x_1) \right| + \Delta h(x) \},$$

for every x_1 such that $|x - x_1| < 2^m$.

Allowing N tend to infinity, in (48b), we obtain (45).

Formula (49) shows that $\overline{M}_N$ satisfies the condition (D) of Corollary 2 of the precedent paper [8], and by theorem IV the operators $\overline{H}_N$ are uniformly bounded on L_p and satisfy uniformly condition (I). Hence by the corollary 2 of the precedent paper the operators $\overline{M}_N$ and $\overline{M}$ are uniformly of the type p , $p > 1$, and satisfy uniformly condition (I), on the set $\mathbf{D}(R^n)$ of the step functions of R^n . By Proposition I and Corollary I, it follows that $\overline{M}$ and $\overline{M}_N$ are uni-

formly, of the type p , for $p > 1$, and satisfy condition (1). From this we deduce by the argument used in theorem III and IV, that Mf is of the type p for $p > 1$, and satisfies condition (1). This proves the theorem.

Now we know that M is of the type p , for $p > 1$, and satisfies condition (1), and that $H_m f(P)$ converges to $Hf(P)$ almost everywhere for every $f \in L^2(\Omega, \mu)$. It is easy to see that the operators H_m satisfy condition B) of § 2. Besides using lemma 1 and lemma 4, and condition z), we will have:

$$\|H_m f\|_p \leq O_p \left\| \sum_{i=1}^m K_i \right\|_1 \cdot \|f\|_p,$$

so that, if m is fixed, $\|f\|_p \rightarrow 0$ implies $\|H_m f\|_p \rightarrow 0$.

Hence, by the argument used in Proposition I, b), we obtain the following theorem

THEOREM VI. *The operators $H_m f$ converge in the p^{th} mean to Hf for every $f \in L^p$, and every $p > 1$. $H_m f(P)$ converges to $Hf(P)$ almost everywhere for any $f \in L^p$ and every $p \geq 1$. Finally, for every set $E \subset \Omega$ of finite measure*

$$\lim_{m \rightarrow \infty} \int_E |Hf(P) - H_m f(P)|^p d\mu = 0, \quad \text{if } f \in L^1 \text{ and } 0 < p < 1;$$

$$\lim_{m \rightarrow \infty} \int_E |Hf(P) - H_m f(P)| d\mu = 0, \quad \text{if } |f| (1 + \log^+ |f|) \in L^1$$

The theorem V unifies the Wiener ergodic dominated theorems and the Zygmund maximal theorem for Hilbert transforms, and in particular the inequality of M. Riesz and Kolmogoroff for the L^p -spaces. The theorem VI unifies the ergodic theorems of V. Neumann and Birkhoff and the theorem of Lusin-Privaloff-Plessner-Riesz, for Hilbert transforms for the L^p -spaces, $p \geq 1$.

Finally, using the results of the paper [8] it is easy to extend the theorems V and VI to double transforms $H_m \times H_s$. In fact, consider s euclidean spaces $R^{n_r} = \{x^r\}$, ..., $R^{n_s} = \{x^{(s)}\}$, and in each of these spaces R^{n_r} , $r = 1, \dots, s$, a kernel $K^{(r)}(x^{(r)})$ satisfying conditions z), $\beta)$, $\gamma_1)$ and $\delta)$, and for each $K^{(r)}(x^{(r)})$ we define the sequence of kernels $K_i^{(r)}$ and the operators $H_m^{(r)} f$ by the formulas (7) and (10). Now let us form the product space $\Omega^1 \times \dots \times \Omega^s = \{P^1, \dots, P^{(s)}\}$, and the product operators:

$$\begin{aligned}
 (50) \quad \widetilde{H}_{m_1, m_2, \dots, m_s} f(P^1, \dots, P^s) &= \widetilde{H}_{m_1}^{(1)} \widetilde{H}_{m_2}^{(2)} \dots \widetilde{H}_{m_s}^{(s)} f(P^1, \dots, P^s) \\
 &= \sum_{i_1=-m_1}^{m_1} \dots \sum_{i_s=-m_s}^{m_s} \int_{R^{i_1}} \dots \int_{R^{i_s}} K_{i_1}^{(1)}(t^1) \dots K_{i_s}^{(s)}(t^s) \\
 &\quad f(\sigma_{t_1} P^1, \dots, \sigma_{t_s} P^s) dt^1 \dots dt^s.
 \end{aligned}$$

By (45), these operators satisfy the hypothesis of theorem 3 of the precedent paper [8], therefore from theorem 3 of the precedent paper, Proposition III and theorems IV, V, and VI, and using proposition Ia, we deduce easily the following theorem

THEOREM VII a) *The maximal operator*

$$\widetilde{M}f(P^1, \dots, P^s) = \sup_{m_1, \dots, m_s} \widetilde{H}_{m_1, \dots, m_s} f(P^1, \dots, P^s)$$

is of the type p, for every $p > 1$, and the operators

$\widetilde{H}_{m_1, \dots, m_s} f(P^1, \dots, P^s)$ converge in the p^{th} mean and pointwise for every $f \in L^p(\Omega^1 \times \dots \times \Omega^s, \mu^1 \times \dots \times \mu^s)$, if $p > 1$

b) If the operators (50) are defined on the spaces C^{nr} , instead of R^{nr} and if $\mu^s(\Omega^s) < \infty$, then

$$\left\{ \dots \int_{\Omega_1} \int_{\Omega_s} |\widetilde{M}f|^\alpha dP^1 \dots dP^s \leq O_\alpha \right\} \left\{ \dots \int_{\Omega_1} \int_{\Omega_s} |f| (1 + \log |f|)^{\alpha-1} dP^1 \dots dP^s \right\} + O_\alpha,$$

for every $\alpha < 1$, where $C^{n_1} \times \dots \times C^{n_s}$ is the $(n_1 + \dots + n_s)$ — dimensional torus (cfr. [8]).

In this case the operators $\widetilde{H}_{m_1, \dots, m_s} f(P^1, \dots, P^s)$ converge pointwise to $\widetilde{H}f(P^1, \dots, P^s)$ almost everywhere, for every function f such that $|f| (1 + \log^+ |f|)^{s-1} \in L^1$.

It is necessary to observe that in the last theorem $m_1, \dots, m_s$ tend to infinity independently one of the other.

Theorem VII contains as particular cases the ergodic theorems of Zygmund and Dunford and theorems of Sokolowski and Zygmund, for ordinary double Hilbert transforms, mentioned in the introduction. Sokolowski and Zygmund consider only the double Hilbert transform of the ordinary 1-dimensional kernel $K(t) = t^{-1}$,

and their proofs are based on the theory of analytic functions or trigonometrical series, of two variables. Theorem VII, furnishes, in particular, the extension of Zygmund-Sokolowski's results for the n -dimensional kernels $K(x) = w(x')/|x|^n$, and by purely measure theoretical methods.

6. Final remarks. We want to mention some partial and still incomplete results, which we hope to develop in future papers, and indicate some fundamental problems.

A) As an illustration of the method of functional relations, mentioned at the end of the Introduction, we will show that, in the case of the ordinary 1-dimensional Hilbert transform Hf , there exist simple relations between Hf and f which actually permit a very simple proof for this basic properties of Hf . The method can be extended to more general situations.

Let $\mathbf{D}$ be the set of all step functions defined on $R^n = \{x\}$ (or any other set, dense in all the L^p -spaces), and $H = Hf(x)$ an operator with the following four properties: 1) Hf assigns to each function $f \in \mathbf{D}$ another function $Hf(x)$, finite almost everywhere on R^n , and Hf is linear in f . 2) H is of the type 2 on $\mathbf{D}$, that is $\|Hf\|_2 \leq O_2 \|f\|_2$ for any $f \in \mathbf{D}$. 3) $Hf(x)$ preserves translations and dilatations, that is, $g(x) = f(x+a)$ implies $(Hg)(x) = (Hf)(x+a)$, and $h(x) = f(\lambda x)$ implies $(Hh)(x) = (Hf)(\lambda x)$.

4) for any $f, g \in \mathbf{D}$ we have

$$(51) \quad |(Hf, g)| = \left| \int_{R^n} Hf(x) \cdot g(x) dx \right| = \left| \int_{R^n} Hg(x) \cdot f(x) dx \right| = |(f, Hg)|$$

For instance the Hilbert transform $Hf = \bar{H}f = \lim \bar{H}_m f$ is perfectly well defined on $\mathbf{D}$ and satisfies conditions 1)-4).

PROPOSITION. Let Hf be an operator defined on $\mathbf{D}$ and satisfying conditions 1)-4). If, besides, H satisfies

$$(52) \quad |f(x)|^2 \leq O_1 |f(x)|^2 + O_2 |H[f \cdot Hf](x)|, \quad f \in \mathbf{D},$$

where O_1, O_2 do not depend on f , then H is of the type p , for every $p > 1$.

PROOF. We will first show that if H is of type p_0 , then it is also of the type $2p_0$. In fact, from (52) we have

$$(53) \quad \int |Hf(x)|^{2p_0} dx \leq O_{p_0} \left\{ \int |f(x)|^{2p_0} dx + \int |H(f \cdot Hf)(x)|^{p_0} dx \right\}.$$

If H is of the type p_0 , then

$$\begin{aligned} \int_{R^n} |H(f, Hf)(x)|^{p_0} dx &\leq O_{p_0} \left\{ \int_{R^n} |f(x)|^{p_0} |Hf(x)|^{p_0} dx \right. \\ &\quad \left. \leq O_{p_0} \left\{ \int_{R^n} |f(x)|^{2p_0} dx \right\}^{1/2} \cdot \left\{ \int_{R^n} |Hf(x)|^{2p_0} dx \right\}^{1/2}, \right. \end{aligned}$$

and from (53) we get

$$\begin{aligned} \int_{R^n} |Hf(x)|^{2p_0} dx &\leq O_{p_0} \left\{ \int_{R^n} |f(x)|^{2p_0} dx \right. \\ &\quad \left. + \int_{R^n} |Hf(x)|^{2p_0} dx \right\}^{1/2} \cdot \left\{ \int_{R^n} |f(x)|^{2p_0} dx \right\}^{1/2} \end{aligned}$$

Defining ξ by

$$\int_{R^n} |Hf|^{2p_0} dx = \xi \cdot \int_{R^n} |f|^{2p_0} dx,$$

it follows that ξ satisfies the inequality $\xi \leq O_{p_0} (1 + \xi^{1/2})$, hence $\xi \leq O = O(p_0)$, that is

$$\int_{R^n} |Hf|^{2p_0} dx \leq O_{p_0} \int_{R^n} |f|^{2p_0} dx,$$

and this proves the assertion. Now, since H is, in virtue of 2), of the type 2, it follows that H is of type p for $p = 2, 4, 8, \dots$. From condition 4) it follows then easily that H is also of the type p' for $p' = 4/3, 8/7, 16/15$. Hence (cfr. theorem 1 of the precedent paper [7]), H is of the type p , for every $p > 1$, and this proves the proposition.

We will now deduce the basic properties of the ordinary 1-dimensional Hilbert transform by extremely simple considerations. For any step function f , the Hilbert operator

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(t)}{x-t} dt$$

is well defined. If $f = f_1$ is the characteristic function of an interval $I \subset R^1$ then $Hf(x)$ is calculated very easily (cfr. Introduction 6)). Hence, for the simple case when $f(x)$ is the characteristic function of $(0, 1)$ it is easy to verify the relation:

$$(54) \quad H^{(2)} f(x) = HHf(x) = -f(x).$$

Since this is a linear relation, and since Hf preserves translations and dilatations it follows that (54) is true for all step functions $f \in \mathbf{D}$.

It is easy to see that H satisfies $|(Hf, g)| = |(f, Hg)|$. Therefore from (54) it follows that $(Hf, Hf) = |(f, H(Hf))| = |(f, -f)| = (f, f)$, hence $\|Hf\|_2 = \|f\|_2$, and H is of the type 2 on $\mathbf{D}$. Thus H satisfies the four conditions 1)-4). It is easy to verify that H satisfies the functional relation

$$(55) \quad [Hf(x)]^2 = (f(x))^2 + 2H(f \cdot Hf)(x)$$

for every $f \in \mathbf{D}$. This can be checked by a direct calculation or by the following argument: For the simple case $f = f_t$ it is easy to check that the Fourier transform $\hat{H}f$ of Hf satisfies the relation

$$(55a) \quad \hat{H}f(x) = \hat{f}(x) \cdot \hat{k}(x), \quad \text{where} \quad \hat{k}(x) = i \cdot \text{sig } x$$

Hence by linearity (55a) is true for any $f \in \mathbf{D}$. Since the relation (55) is equivalent to

$$(Hf \star Hf)(x) = (f \star f)(x) + 2k(x) \{ f(x) \star Hf(x) \},$$

formula (55a) shows that this relation is true for any $f \in \mathbf{D}$.

From (55) and the above Proposition, we deduce that H is of the type p , for every $p > 1$.

In the precedent paper ([8], Example A), § 2) we have seen by very simple calculations that if H_+ and M_+ are defined by

$$H_+ f(x) = \int_{|x-t| > 1} \frac{f(t)}{x-t} dt$$

and

$$M_+ f(x) = \sup_{t > 0} |H_+ f(x)|,$$

then M and H are related by the inequality (D) of the precedent paper, and consequently M is of the type p on $\mathbf{D}$, for $p > 1$. We are now in position to apply Proposition I and Corollary I, and we obtain the pointwise and mean convergence of $H_+ f$ to Hf , as well as the Riesz and Zygmund inequalities.

We have thus obtained the proof of the fundamental facts of the ordinary 1-dimensional Hilbert transform by very simple considerations using only the functional relations (54), and (55), which have to be checked only in the simplest case when $f(x)$ is the characteristic function of the interval $(0,1)$, and this is a matter of elementary Analysis.

B) If we are interested only in the mean convergence of the operators H in the L^2 -space, then the assumptions on the kernel $K(x)$ can be considerably weakened. As an example we will consider here the case of the 2-dimensional Hilbert transform kernel $K(x)$ defined by

$$K(z) = \frac{w(\theta)}{|z|^2} \quad \text{if } 1 \leq |z| \leq 2, \quad \text{and zero otherwise,}$$

where $z = |z|e^{i\theta}$ and $\int_0^{2\pi} w(\theta) d\theta = 0$.

Let

$$K_{\lambda, \mu}(z) = \frac{w(\theta)}{|z|^2} \quad \text{if } 0 < \lambda \leq |z| \leq \mu,$$

and zero otherwise,

$$W(\theta) = \int_0^\theta w(t) dt,$$

and

$$V_\epsilon(\theta) = \int_{-\pi/2}^{\pi/2} \frac{W(\theta+t) - W(\theta-t)}{t} dt = O\left(\int_{-\pi/2}^{\pi/2} \frac{W(\theta+t) - W(\theta-t)}{tg t} dt\right)$$

Since V_ϵ is the «cutted» Hilbert transform defined on $(0, \pi)$ of the function W , and since W is absolutely continuous, the limit

$$\lim_{\epsilon \rightarrow 0} V_\epsilon(\theta) = V(\theta),$$

exists for almost all θ .

In the following lemma $w(\theta)$ is only assumed to be integrable and have its integral equal to zero. By $\hat{K}(u)$ we denote the Fourier transform of $K(x)$.

LEMMA. Let $z = |z|e^{i\theta}$ and $u = |u|e^{i\phi}$. Then, for almost all σ and u it is true that :

$$a) \quad \hat{K}_{\lambda, \mu}(u) = R_{\lambda, \mu} + V_{\frac{1}{\mu|u|}}\left(\sigma + \frac{\pi}{2}\right) + V_{\frac{1}{\lambda|u|}}\left(\sigma + \frac{3\pi}{2}\right),$$

where $|R_{\lambda, \mu}(u)| \leq O_1 = \text{absolute constant}$.

b) For almost all u there exists the limit

$$\hat{K}_\infty(u) = \lim_{\lambda \rightarrow 0, \mu \rightarrow \infty} \hat{K}_{\lambda, \mu}(u) = V\left(\sigma + \frac{\pi}{2}\right) + V\left(\sigma + \frac{3\pi}{2}\right).$$

$\hat{K}_\infty(u)$ is independent of $|u|$, that is, it depends only on σ , and as a function of σ :

$$\hat{K}_\infty \in L^2(0, 2\pi) \quad \text{and} \quad \|\hat{K}_\infty\|_2 \leq O_2 \cdot \|W\|_2.$$

c) If $|\hat{K}_\infty(u)| \leq M$, then

$$|\hat{K}_{\lambda, \mu}(u)| \leq O_1 \{ M + \|W\|_2 + \|w\|_2 \}.$$

PROOF. We have

$$\begin{aligned} \hat{K}_{\lambda, \mu}(u) &= \int_0^{2\pi} \left\{ w(\sigma + \theta) \int_{\lambda|u|\cos\theta}^{\mu|u|\cos\theta} \frac{e^{i\rho}}{\rho} d\rho \right\} d\theta \\ &= \int_0^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi} w(\sigma + \theta) \int_{\lambda|u|\cos\theta}^{\mu|u|\cos\theta} \frac{e^{i\rho}}{\rho} d\rho = I_1 + I_2 + I_3 + I_4. \end{aligned}$$

The first integral may be written

$$\begin{aligned} I_1 &= \int_0^{\pi/2} w(\sigma + \theta) \int_{\lambda|u|\cos\theta}^{\mu|u|\cos\theta} \frac{e^{i\rho}-1}{\rho} d\rho + \int_0^{\theta_0} w(\sigma + \theta) \int_{\lambda|u|\cos\theta}^1 \frac{e^{-i\rho}-1}{\rho} d\rho \\ &\quad + \int_1^{\mu|u|\cos\theta} \frac{e^{i\rho}}{\rho} d\rho + \int_{\theta_0}^{\pi/2} w(\sigma + \theta) \int_{\lambda|u|\cos\theta}^{\mu|u|\cos\theta} \frac{d\rho}{\rho} + \int_{\theta_0}^{\pi/2} w(\theta + \sigma) \int_{\lambda|u|\cos\theta}^1 \frac{d\rho}{\rho}. \end{aligned}$$

Here θ_0 is such that $\mu|u|\cos\theta_0=1$, or $\mu|u|\cos(\pi-\theta_0)=1$.
Taking in account that

$$\left| \int_1^{\mu|u|\cos\theta} \frac{e^{i\rho}}{\rho} d\rho \right| \leq O_1,$$

we see that the first three integrals give a function $R^1_{\lambda, \mu}(u)$ such that $|R^1_{\lambda, \mu}(u)| \leq O_1$ so that

$$\begin{aligned} I_1 &= R^1_{\lambda, \mu}(u) - \log \lambda |u| \int_0^{\pi/2} w(\theta) d\theta + \log \mu |u| \left[W\left(\frac{\pi}{2}\right) - W(\theta_0) \right] \\ &\quad - \int_0^{\theta_0} w(\sigma + \theta) \log \cos \theta d\theta. \end{aligned}$$

and integrating by parts the last term,

$$I_1 = R_{\varepsilon_0}^1(u) - \log \lambda \left[u \right] \int_{\theta_0}^{\pi/2} w(\theta) d\theta + (\log \mu \left[u \right]) W\left(\frac{\pi}{2}\right) - \int_{\theta_0}^{\pi/2} \frac{W(\theta) \sin \theta}{\cos \theta} d\theta.$$

Applying similar transformations to I_2, I_3, I_4 , we get

$$\begin{aligned} \hat{K}_{\varepsilon_0}(u) = & R_{\varepsilon_0}^1 + R_{\varepsilon_0}^2 + R_{\varepsilon_0}^3 + R_{\varepsilon_0}^4 = \\ & \int_{\theta_0}^{\theta_1} + \int_{\pi-\theta_0}^{\pi} + \int_{\pi/2-\theta_0}^{\pi/2} \frac{w(\sigma+\theta) \sin \theta}{\cos \theta} d\theta + \int_{\pi/2-\theta_0}^{\pi/2} w(\theta) \frac{\sin \theta}{\cos \theta} d\theta. \end{aligned}$$

Since $(\pi/2 - \theta_0)^{-1} = O[(\sin(\pi/2 - \theta_0))^{-1}] = O[(\cos \theta_0)^{-1}]$, and since $[\cos \theta_0]^{-1} = \mu \left[u \right]$, we obtain a).

From the proof of a) is also clear that $\lim \hat{K}_{\varepsilon_0}(u) = V(\sigma + \pi/2) + V(\sigma + 3\pi/2)$, and since $\|V\|_2 \leq O_2 \|W\|_2$, we obtain b). Finally if $|\hat{K}_{\infty}(u)| \leq M$, we obtain from b) that $|V(\sigma + \pi/2) + V(\sigma + 3\pi/2)| \leq M$. Therefore the Abel sums of the Fourier series of $V(\sigma + \pi/2) + V(\sigma + 3\pi/2)$ —conjugated Fourier series of W , are uniformly bounded by $O_1 M$, and being bounded, the $V\left(\frac{1}{\lambda \left[u \right]}\right), V\left(\frac{1}{\mu \left[u \right]}\right)$ differ from the Abel sums in less than $O_1 \sup |W(\theta)|$, and we obtain c). This proves the lemma:

COROLLARY. Let $K^{(n)}(z) = w^{(n)}(\theta) [z]^2$ if $1 \leq |z| \leq 2$, and zero otherwise, be a sequence of kernels. If $\lim_{n \rightarrow \infty} \sup_{|z| \leq 2} |W(\theta) - W^{(n)}(\theta)| = 0$, then $\hat{K}_{\infty}^{(n)}(u) = \hat{K}_{\infty}(u)$ almost everywhere.

Assume now that the Fourier series of $w(\theta)$,

$$w(\theta) \sim \sum \alpha_n e^{in\theta},$$

is such that

$$\sum \beta_n e^{in\theta} = \sum \varepsilon_n \frac{\alpha_n}{n} e^{in\theta}$$

is bounded ($\varepsilon_n = \pm 1$). Then it follows from the lemma, and from well known properties of the Fourier series, that $|K(u)| \leq M$ and $|w| \leq M$. Since $\bar{H}_m f = f * K_{\varepsilon_m}$, $\bar{H}_m f(u) = f(u) * \bar{K}_{\varepsilon_m}(u)$, we obtain $\|\bar{H}_m f\|_2 \leq \|f\|_2 \cdot M$, and by the Plancherel it follows that the operators $\bar{H}_m f$ are uniformly bounded

in L^2 . Moreover, from $K_{n,n}(u) \rightarrow K_\infty(u)$, boundedly, it follows the mean converge of $H_n f$, for every $f \in L^2$. Furthermore, from the Corollary, it follows that if the operators $\tilde{H}^m f$ are defined by the kernels

$$K^{(m)} = \sum_{K=-n}^n \alpha_K e^{iK\theta},$$

given by the partial sums of the Fourier series of $w(\theta)$, then $\tilde{H}^{(m)} \rightarrow \tilde{H}$ (cfr. Mihlin [14]). Similar generalizations may be obtained for the general kernels and operators H_m defined in § 1, using the Hardy-Littlewood tauberian theorem.

C) We will mention now two fundamental problems for the general operators H_m . The theorem VII proves the convergence of the double transforms $\tilde{H}_{m,m} f(P_1, P_2)$, and the boundedness of $\tilde{M} f(P_1, P_2)$, only for $f \in L^p$ with $p > 1$. It is an open problem (even for the ordinary double Hilbert transform) whether the maximal operator $\tilde{M}$ satisfies condition (1), and whether the pointwise convergence is true for $f \in L^1$. This problem is specially important, because if $\tilde{H}_{m,m} f$ is the ordinary double Hilbert transform on $C^1 \times C^1$ it reduces to the problem of the Abel summability of double trigonometrical series.

In the particular case where $\tilde{H}_m f = D_\varepsilon f$, $\varepsilon = m^{-1}$, theorem VII reduces to known properties of the theory of strong differentiability. In this case it is known that $\lim D_\varepsilon D_\eta f(x, y)$ exists if ε and η tend to zero independently and $|f| \log^+ |f| \in L^1$, or if $\varepsilon = \eta$ and $f \in L^1$.

Since the operators $D_\varepsilon D_\eta$ of the theory of strong differentiability are very particular cases of the operators $\tilde{H}_{m,m}$, and since the theory of strong differentiation has been largely developed, several important problems arise for the general double operators $\tilde{H}_{m,m}$. The most important of them are the following.

In analogy with the theory of strong differentiation it is natural to conjecture that the limit of $H_{m,m} f(x_1, x_2)$, $x_1 \in R^1_1$, $x_2 \in R^1_2$, does not exist, and M does not satisfy condition (1), if $f \in L^1$, unless the indefinite integral of $f(x_1, x_2)$ is strongly differentiable. Instead, it is natural to conjecture that the limit of $\tilde{H}_{m,m}$ exists if $m_1 = m_2$ and $f \in L^1$, or if $m_1 \neq m_2$ and $|f| \log^+ |f| \in L^1$. These problems are still more difficult if $H_{m,m}$ is defined on a general space $\Omega_1 \times \Omega_2$ because it is necessary first to extend the theorems of the theory of strong

differentiation to the operators $\mathbf{D}_N$ of the ergodic theory. We have partially investigated, for these operators $\mathbf{D}_N$ some analogues of the theorems of Ward and Besicovitch, as well as the convergence of $\mathbf{D}_N \mathbf{D}_N f$ (even if σ_x, σ_x' do not commute) (cfr. [19]), however our results are still too incomplete to be given here.

In analogy with the classical theory of differentiation of functions of intervals $F(J)$, it could be also interesting to investigate the following kind of problems. Let Ω, σ_x , be as before, and consider a function $F(P, J)$, where $P \in \Omega$, and J is an interval (or a cube) of R^1 , such that:

- (i) $F(P, J) = F_J(P)$ is a measurable function in P , for fixed J .
- (ii) $F(P, J) = F_J(J)$ is an additive function of the set J .
- (iii) $F(\sigma_t P, J) = F(P, J+t)$.

If a is a fixed real number we define

$$\mathbf{D} F(P, a) = \limsup_{|J| \rightarrow 0} \frac{F(P, J)}{|J|}, \quad a \in J, \quad |J| \rightarrow 0,$$

and similarly we define the derivatives $\mathbf{D} F(P, a)$ and $\mathbf{D} F(P, \infty)$. For instance, if $f \in L^1(\Omega)$ and if

$$F(P, J) = \int_a^b f(\sigma_t P) dt = \int_J f(\sigma_t P) dt, \quad J = (a, b),$$

then the ergodic theorem states that the derivatives $\mathbf{D} F(P, \infty)$ exist for almost all P . It seems that theorems of unicity of the Perron-Denjoy theory, as well as other facts from the theory of differentiation, could be extended to these general derivatives.

D) The limit operators $Hf = \lim H_n f$ are continuous, but not completely continuous, operators on the Hilbert space $L^2(\Omega)$. While the classical theory of Fredholm-Riesz studies equations of the form $(I - T)f = g$ where I is the identity operator and T is a completely continuous operator, Carleman, Noether, Tricomi, Mijlin and other have studied the singular equation of the form $hf + \bar{H}f + Tf = g$, where $\bar{H}$ is the n -dimensional Hilbert transform and T a completely continuous operator (cfr. Mikhlin [14]). These equations correspond to the general theory of quasi-Fredholm operators, developed by Nikolski, Jalilov and Atkinson (cfr. [20]).

Thus, the first fundamental problem for the general transformations $Hf = \lim H_m f$, is to investigate whether the Tricomi-Mikhlin theory applies to these operators Hf , and whether these operators H give all the quasi-Fredholm operators on $L^2(\Omega)$.

For this purpose it is necessary first to investigate the spectral properties of the operators H and specially extend the theory of « symbols » of Tricomi-Giran-Mikhlin. Finally, it seems natural to investigate the operators of the form $H_m f = f \star T_m$ where T_m are Schwartz's distributions « generated » by a fixed distribution.

We conjecture that the Hilbert transforms may be characterized as those operators $Hf = \lim H_m f = \lim (f \star T_m)$ which possess the two following properties: they are quasi Fredholm operators, and second, they commute with translations and dilatations (cfr. § 1; G). On the other hand the operators of the form $H_m T = T \star \sum_{m=1}^{\infty} K_i$ should be considered, where T is a distribution. Such operators have been considered by Horvath [21] for the particular kernels $K(x) = x^{-1}$.

The solution of this fundamental problem would show which of the generalized operators $Hf(P) = \lim H_m f(P)$ defined in § 1, could be of real interest for the Analysis.

RESUMEN

La analogía entre la teoría de transformadas de Hilbert, la teoría de derivación y la teoría ergódica, ha sido recalada por varios autores (Lusin, Zygmund y otros). Sin embargo estas teorías han sido tratadas por métodos enteramente diferentes, siendo las demostraciones considerablemente más complejas en el caso de las transformadas de Hilbert. Esto se debe al hecho que la teoría ergódica considera operadores positivos, mientras que las transformadas de Hilbert son operadores no-positivos.

El objeto de este trabajo es dar una teoría general que contiene como casos particulares a las teorías mencionadas. Sea $R^n = \{x\}$ el espacio euclídeo n -dimensional, $K(x)$ una función integrable, y sea $K_i(x) = 2^{-ni} K(2^{-i}x)$, $i = \pm 1, \pm 2, \dots$. Sea, por otra parte $\Omega = \{P\}$ un espacio abstracto con una medida μ y con un grupo n -dimensional $\{\tau_x\}$, $x \in R^n$, de transformaciones isomédicas. Es decir, para cada $x \in R^n$, τ_x es una transformación de Ω en Ω que transforma conjuntos medibles en medibles, conservando la medida. Para cada $m = 1, 2, \dots$ definimos el operador

$$H_m f(P) = \sum_{i=-m}^m \int_{R^n} f(\tau_x P) K_i(x) dx.$$

Bajo ciertas hipótesis sobre el núcleo $K(x)$, probamos los resultados siguientes:

- 1) La sucesión $H_m f(P)$ converge puntualmente hacia un límite $Hf(P)$, para

toda $f \in L_p(\Omega, \mu)$, y para todo $p \geq 1$. 2) Si $p > 1$, entonces $H_m f$ converge también en media- p . 3) El operador maximal $Mf(P) = \sup_m |H_m f(P)|$ es acotado sobre la esfera unitaria de $L_p(\Omega, \mu)$, si $p \geq 1$.

Si $\Omega = \mathbb{R}^n = \{t\}$, $\tau_h t = t+h$, y si $K(x) = w(x) |x|^{-n}$ para $1 \leq |x| \leq 2$, $K(x) = 0$ en los demás puntos x , $w(x)$ es una función que no depende de $|x|$, entonces $Hf = \lim H_m f$ es la transformada de Hilbert n -dimensional. En este caso, 1), 2) y 3) dan los resultados clásicos de Lusin-Riesz-Kolmogoroff (ver [1] en la bibliografía que precede), si $n = 1$, y los resultados recientes de Zygmund y Calderón ([2]), si $n > 1$.

Si Ω es un espacio general, y si $K(x) = -1$ para $|x| < 1$, $K(x) = +1$ para $1 \leq |x| \leq 2$, $K(x) = 0$ en los demás puntos, entonces H_m son los operadores ergódicos y 1), 2), 3) dan los teoremas ergódicos de v. Neumann, Birkhoff y Wiener ([3]).

Probamos además resultados similares para los operadores dobles $H_m f(P, Q)$. De estos se obtienen, como casos particulares, los teoremas ergódicos n -paramétricos de Zygmund-Dunford ([4]), así como los teoremas de Zygmund-Sokolowsky ([5]) referentes a la transformada doble de Hilbert, 1-dimensional. Nuestros resultados proporcionan en particular una extensión de estos resultados de Zygmund-Sokolowsky a núcleos n -dimensionales.

En las demostraciones de los teoremas anteriores usamos tan solo métodos de la teoría de la medida. En efecto, nuestras demostraciones se basan en tres teoremas sobre espacios L^p , que han sido dados en los tres trabajos precedentes ([6], [7], [8]). Si estamos interesados solo en el caso $\Omega = \mathbb{R}^n$, entonces la teoría se simplifica considerablemente, como ha sido indicado en los ejemplos de los tres trabajos mencionados; esto proporciona, en particular, un tratamiento nuevo y simplificado de la teoría de transformadas de Hilbert.

Por otra parte, mostramos que usando teoremas tauberianos en grupos topológicos, se puede extender la teoría para núcleos $K(x)$ definidos en grupos abelianos localmente compactos. Entre otras cosas, esto permite la unificación de las teorías «discretas» y «continuas», tales como la teoría de M. Riesz de transformadas de Hilbert discretas o de sucesiones y la teoría ordinaria de transformadas de Hilbert. Otra aplicación, es una extensión del teorema ergódico a grupos generales, debido a Calderón [9]. Desde nuestro punto de vista, las transformadas de Hilbert y los teoremas ergódicos deberían obtenerse como casos particulares de teoremas tauberianos generales.

En el trabajo presente extendemos a los operadores generales H_m tan solo los hechos más fundamentales de la teoría de transformadas de Hilbert o de la teoría ergódica. Nos parece que la extensión de los aspectos más profundos de estas teorías podría abrir un campo interesante para la investigación (ver especialmente § 6 y D).

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CONTENIDO

- MISCHA COTLAR, A combinatorial inequality and its applications to L^2 -spaces.
MISCHA COTLAR, A general interpolation theorem for linear operations.
MISCHA COTLAR, Some generalizations of the Hardy-Littlewood maximal theorem.
MISCHA COTLAR, A unified theory of Hilbert transforms and ergodic theorems.